Indonesian Journal of Statistics and Its Applications
DOI: https://doi.org/10.29244/ijsa.v10i1p60-78

RESEARCH ARTICLE

Asymmetric Laplace Stochastic Volatility Model and its Applications

Rahul Thekkedath[1], Shiji Kavungal[2]

[1]Department of Statistics and Information Management, Reserve Bank of India, India
[2]Department of Statistics, Sree Kerala Varma College, Thrissur, Kerala, India

OPEN ACCESS

Check for updates

Article History
Received 07 May 2026
Revised 15 June 2026
Accepted 20 June 2026

Keywords
Asymmetric Laplace Distribution, Autoregressive Process, Financial Time Series, Stochastic Volatility

Corresponding Author:
Rahul Thekkedath

Email:
rahult@rbi.org.in

Department of Statistics and Information Management, Reserve Bank of India, Thiruvananthapuram, Kerala, India.

ABSTRACT
This paper proposes a stochastic volatility model driven by a first-order autoregressive process with an asymmetric Laplace marginal distribution. The novelty of the proposed model lies in incorporating an autoregressive structure with asymmetric Laplace marginal into the log-volatility equation, thereby providing a flexible framework for capturing asymmetry, heavy tails, and leptokurtic behavior commonly observed in financial return series. Parameter estimation is carried out using the generalized method of moments, which avoids computationally intensive likelihood evaluations and provides an efficient alternative to conventional estimation procedures for stochastic volatility models. A simulation study is conducted to examine the finite-sample performance of the proposed estimators under different parameter settings and sample sizes. The results indicate that the estimators perform satisfactorily, exhibiting small bias and improved accuracy as the sample size increases. The practical usefulness of the proposed model is illustrated through applications to daily exchange rate returns and crude oil futures returns. The empirical findings demonstrate that the proposed model adequately captures important stylized facts of financial returns, including skewness, excess kurtosis, and volatility clustering. Overall, the asymmetric Laplace stochastic volatility model offers a flexible and computationally attractive alternative to existing stochastic volatility models based on Gaussian log-volatility dynamics.

Introduction

It is well known that the return distributions of financial time series—such as stock and foreign exchange returns—often deviate from the normality assumption, as they tend to be skewed and exhibit heavier tails than the normal distribution. These data also display time-varying volatility and volatility clustering. Accounting for these features is crucial for making sound risk management decisions. To address these characteristics, researchers have developed two main classes of models that capture autocorrelated time-varying volatility: the autoregressive conditional heteroscedasticity (ARCH) model, introduced by Engle (1982), and the stochastic volatility (SV) model, introduced by Taylor (1986). In the ARCH model, the time-varying variance is assumed to be a deterministic function of the lagged squared errors. In contrast, the SV model treats volatility at time t as a stochastic process driven by latent variables. For a comprehensive survey of volatility models and their various generalizations—such as the generalized ARCH (GARCH) model proposed by Bollerslev (1986)—see Tsay (2005), Bauwens et al. (2012) and So et al. (2022).

The basic SV model with a conditional normal distribution for the returns is too restrictive to account for the usual asymmetry and heavier tails observed in financial return series. Many studies have been concerned with modelling the conditional return with distributions which have heavier tails than those of normal distribution. One of the most popular

© 2026 Thekkedath et al. This is an open-access article distributed under the terms of the Creative Commons Attribution (CC BY) license, allowing unrestricted use, distribution, and reproduction in any medium, provided proper credit is given to the original authors.
Print only if necessary. Let's protect our trees and the planet together.

choices for return distributions is the Student's t distribution (See Chib et al., 2002; Omori et al., 2007; Choy et al., 2008; Nakajima and Omori, 2009). Tsiotas (2012) proposed the SV models with the asymmetric non-central t and normal distributions. Based on the generalized hyperbolic skew t distribution of Aas and Haff (2006), Nakajima and Omori (2012) extended the model with the t distribution such that the skewness is taken into account. Their model was further extended by Nakajima (2013) such that the skewness is allowed vary over time through the Markov switching process. The other studies on non-normal SV models include the one with Normal Inverse Gaussian distribution by Barndorff-Nielson (1997) and Anderson (2001); generalized error distribution by Liesenfeld and Jung (2000); generalized Student's t distribution by Wang et al. (2013); and so on.

A more straight forward way to capture the non-normality of return series is to use an non-Gaussian first order autoregressive (AR(1)) model for non-negative random variables to generate the volatility sequence. The standard SV model in the literature assumes a Gaussian AR(1) model for generating the volatility sequence. A number of such AR models with non-Gaussian marginal distributions are introduced for modelling non-negative random variables in the context of time series analysis. See, for example, first order AR processes with gamma (Gaver and Lewis, 1980; Lawrance and Lewis, 1985; Sim, 1990; Adke and Balakrishna, 1992); inverse Gaussian (Abraham and Balakrishna, 1999); Cauchy (Balakrishna and Nampoothiri, 2003); normal-Laplace (Jose et al., 2008); approximated beta distribution (Popovic, 2010; Popovic et al., 2010); extreme value (Balakrishna and Shiji, 2014a); Birnbaum-Saunders (Rahul et al., 2018) marginal distributions have been discussed in the literature. For a comprehensive review of non-Gaussian autoregressive type time series models and its properties see Balakrishna (2021) and the references cited therein. In principle, one can very well use these AR models to describe the evolution of time-dependent volatilities. For example, as an alternative to normal-lognormal SV models, Abraham et al. (2006) proposed a SV model generated by a gamma AR(1) sequence of Gaver and Lewis (1980) and Balakrishna and Shiji (2014b) developed a SV model generated by first order Gumbel extreme value autoregressive process. Balakrishna and Rahul (2019) developed a SV with Birnbaum-Saunders Markov sequence of Rahul et al. (2018) and Anvar et al. (2019) analyzed the SV models induced by non-negative Markov sequences generated by generalized gamma product autoregressive models. Recently, Shiji and Rahul (2022) and Sujith and Balakrishna (2023) considered the first order normal-Laplace and inverse Gaussian Markov sequences respectively for generating non-negative volatility in SV model.

The asymmetric Laplace distribution demonstrates flexibility in fitting data with heavy tails and skewness, which make it a promising candidate for financial data modelling. Kozubowski and Podgorski (2000) and Kotz et al. (2001) studied the properties of asymmetric Laplace distributions. Jayakumar and Kuttykrishnan (2006) introduced a time series model using an asymmetric Laplace distribution for modelling data from financial contexts. Jose and Thomas (2011) developed a first order stationary autoregressive process with generalized Laplace marginal distribution. Although the theory and applications of asymmetric Laplace distributions is well developed and there is considerable literature in recent years, their applications in modelling stochastic volatility in financial time series are not developed. In this article, we focus on the suitability of asymmetric Laplace distribution to capture the stylized facts of financial volatility and the properties of asymmetric Laplace SV (AL-SV) models are discussed.

The next section briefly discusses the asymmetric Laplace distribution and its properties. A first order autoregressive model with asymmetric Laplace marginal distribution is constructed in Section 3. The proposed AL-SV model and its second order properties are described in Section 4. We discussed the estimation procedure by the method of moments and the asymptotic properties of estimators in Section 5. A simulation study is carried out in Section 6. In Section 7, we present the results on data analysis using our model.

Asymmetric Laplace Distribution

A random variable X is said to have an asymmetric Laplace distribution with parameters $\boldsymbol{\theta} \in \mathbb{R}$, $\kappa > 0$ and $\sigma \geq 0$ $\left(AL(\theta, \kappa, \sigma)\right)$ if its probability density function is

$$f(x;\theta,\kappa,\sigma) = \frac{\sqrt{2}}{\sigma}\frac{\kappa}{1+\kappa^2}\begin{cases}\exp\left\{\frac{-\sqrt{2}\kappa}{\sigma}|x-\theta|\right\}, if\ x \geq \theta \\ \exp\left\{-\frac{\sqrt{2}}{\kappa\sigma}|x-\theta|\right\}, if\ x < \theta\end{cases} \quad (1)$$

or, the distribution function of the X is the form

$$F(x;\theta,\kappa,\sigma)=\begin{cases}1-\dfrac{1}{1+\kappa^2}\exp\left\{\dfrac{-\sqrt{2}\kappa}{\sigma}|x-\theta|\right\}, if\ x\geq\theta\\ \dfrac{\kappa^2}{1+\kappa^2}\exp\left\{\dfrac{\sqrt{2}}{\kappa\sigma}|x-\theta|\right\}, if\ x<\theta\end{cases} \quad (2)$$

The parameter θ is the location parameter, determining the mode and shifting the entire distribution left or right without affecting its shape. The parameter $\sigma\geq 0$ is the scale parameter, controlling the overall spread of the distribution; larger values of σ produce a wider distribution with greater variability, while smaller values concentrate observations more closely around θ. The parameter $\kappa>0$ is the asymmetry (skewness) parameter, governing the relative decay rates of the left and right tails. When $\kappa=1$, the distribution reduces to the symmetric Laplace distribution. Values $\kappa<1$ produce positive (right) skewness with a longer right tail, whereas $\kappa>1$ produce negative (left) skewness with a longer left tail. Together, these parameters provide a flexible model that can simultaneously capture location, dispersion, and asymmetry while retaining the characteristic sharp peak and exponential tails of the Laplace family.

Hence the characteristic function of $AL(\theta,\kappa,\sigma)$ is obtained as

$$\psi_{X(t)}=E(e^{itX})=\frac{e^{i\theta t}}{1+\frac{1}{2}\sigma^2t^2-i\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)t},\frac{\sqrt{2}}{\sigma\kappa}<t<\frac{\sqrt{2\kappa}}{\sigma} \quad (3)$$

Using (3), the mean, variance and the coefficients of skewness and kurtosis can be respectively obtained as

$$E(X)=\theta+\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right),V(X)=\frac{\sigma^2}{2}\left(\frac{1}{\kappa^2}+\kappa^2\right),\gamma_1=2\left(\frac{\frac{1}{\kappa^3}-\kappa^3}{\left(\frac{1}{\kappa^2}+\kappa^2\right)^{\frac{3}{2}}}\right),\gamma_2=6-\frac{12}{\left(\frac{1}{\kappa^2}+\kappa^2\right)^2} \quad (4)$$

It follows from (4) that the absolute value of γ_1 is bounded by two, and as κ increases within the interval $(0,\infty)$, then the corresponding value of γ_1 decreases monotonically from 2 to -2. Similarly, the distribution is leptokurtic and γ_2 varies from 3 (the least value for the symmetric Laplace distribution with $\kappa=1$) to 6 (the greatest value attained for the limiting exponential distribution when $\kappa\to 0$.

The asymmetric Laplace random variable X also admits the representation of the form

$$X^d=\ \rho X+(1-\rho)\theta+\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}I_1E_1-\kappa\, I_2E_2\right),\rho\in[0,1] \quad (5)$$

where I_1,I_2 are dependent Bernoulli random variables taking on values of either zero or one with probabilities,

$$\mathrm{P}(\mathrm{I}_1=0,\mathrm{I}_2=0)=\rho^2,\mathrm{P}(\mathrm{I}_1=1,\mathrm{I}_2=1)=0 \quad (5a)$$

$$\mathrm{P}(\mathrm{I}_1=1,\mathrm{I}_2=0)=(1-\rho)\left(\rho+\frac{1-\rho}{1+\kappa^2}\right) \quad (5b)$$

$$\mathrm{P}(\mathrm{I}_1=0,\mathrm{I}_2=1)=(1-\rho)\left(\rho+\frac{(1-\rho)\kappa^2}{1+\kappa^2}\right) \quad (5c)$$

E_1 and E_2 are standard exponential variables, with all variables being mutually independent.

As shown in Kotz et al. (2001), all asymmetric Laplace laws are self-decomposable for all values of the parameters. Gaver and Lewis (1980) proved that only self-decomposable distributions can be marginal distributions of a first order autoregressive process. Hence the asymmetric Laplace distribution can be the marginal distribution of an AR(1) process. Using the results in Jose and Thomas (2011), a first order autoregressive process with asymmetric Laplace distribution is constructed in the next section.

First Order Asymmetric Laplace Autoregressive Process

The first order asymmetric Laplace autoregressive process is constituted by $\{h_t, t \geq 1\}$, where h_t satisfies the equation,

$$h_t = \rho h_{t-1} + \eta_t, \rho \in [0,1], t > 0 \tag{6}$$

where $\{h_t\}$ is a stationary Markov process with asymmetric Laplace marginal distribution with location parameter θ, shape parameter κ and scale parameter σ $\left(AL(\theta,\kappa,\sigma)\right)$ as given in (1) and $\{\eta_t\}$ is an independent and identically distributed random variables independent of $h_{t-\tau}$ for all $\tau \geq 1$. The basic problem is to find the distribution of $\{\eta_t\}$ such that $\{h_t\}$ has the asymmetric Laplace distribution as the stationary marginal distribution. The following theorem summarizes the result in this context.

Theorem: The stationary marginal distribution of $\{h_t\}$ in model (6) is asymmetric Laplace marginal distribution with parameters θ, κ and σ if and only if the distribution of η_t is specified as a convolution of the form $\eta_t \overset{d}{=} U + (I_1 E_1 - I_2 E_2)$ as in (8) provided

$$\eta_0 =^{\mathrm{d}} h_0 \tag{7}$$

Proof:

In terms of characteristic function, the model in (6) can be rewritten as

$\psi_h(t) = \psi_h(\rho t)\psi_\eta(t)$.

Under stationarity assumption, $\psi_\eta(t) = \dfrac{\psi_h(t)}{\psi_h(\rho t)}$.

Substituting the characteristic function given by (3) we get,

$$\psi_\eta(t) = \frac{e^{i\theta t}}{e^{i\theta\rho t}} \frac{\left(1 + i\frac{\sigma\kappa}{\sqrt{2}}\rho t\right)\left(1 - i\frac{\sigma}{\sqrt{2}\kappa}\rho t\right)}{\left(1 + i\frac{\sigma\kappa}{\sqrt{2}} t\right)\left(1 - i\frac{\sigma}{\sqrt{2}\kappa} t\right)}$$

$$= e^{i\theta(1-\rho)t}\left[\rho + (1-\rho)\frac{1}{\left(1 + i\frac{\sigma\kappa}{\sqrt{2}} t\right)}\right]\left[\rho + (1-\rho)\frac{1}{\left(1 - i\frac{\sigma}{\sqrt{2}\kappa} t\right)}\right]$$

This implies that η_t has a convolution structure of the following form

$$\eta_t =^{\mathrm{d}} U + (I_1 E_1 - I_2 E_2) \tag{8}$$

where U is degenerate at $\theta(1-\rho)$ and E_1 and E_2 are exponential random variables with means $\sigma/\sqrt{2}\kappa$ and $\sigma\kappa/\sqrt{2}$ respectively and (I_1, I_2) is such that

$$\mathrm{P}(\mathrm{I}_1 = 0, \mathrm{I}_2 = 0) = \rho^2, \mathrm{P}(\mathrm{I}_1 = 1, \mathrm{I}_2 = 1) = (1-\rho)^2 \tag{8a}$$

$$\mathrm{P}(\mathrm{I}_1 = 1, \mathrm{I}_2 = 0) = \mathrm{P}(\mathrm{I}_1 = 0, \mathrm{I}_2 = 1) = \rho(1-\rho) \tag{8b}$$

Further (I_1, I_2) is independent of E_1 and E_2.

The converse part can be provided by the method of induction. We assume that h_{t-1} follows $AL(\theta,\kappa,\sigma)$ with characteristic function given in (3).

$$\psi_{h_t}(t) = \psi_{h_{t-1}}(\rho t)\psi_{\eta_t}(t) \tag{9}$$

$$= \left[\frac{e^{i\theta\rho t}}{\left(1+i\frac{\sigma\kappa}{\sqrt{2}}\rho t\right)\left(1-i\frac{\sigma}{\sqrt{2}\kappa}\rho t\right)}\right]\left[\frac{e^{i\theta(1-\rho)t}\left(1+i\frac{\sigma\kappa}{\sqrt{2}}\rho t\right)\left(1-i\frac{\sigma}{\sqrt{2}\kappa}\rho t\right)}{\left(1+i\frac{\sigma\kappa}{\sqrt{2}}t\right)\left(1-i\frac{\sigma}{\sqrt{2}\kappa}t\right)}\right]$$

$$= e^{i\theta t}\left(\frac{1}{1+i\frac{\sigma\kappa}{\sqrt{2}}t}\right)\left(\frac{1}{1-i\frac{\sigma}{\sqrt{2}\kappa}t}\right),$$

which is same as the asymmetric Laplace characteristic function. This shows that $\{h_t\}$ is strictly stationary with asymmetric Laplace marginals provided $\eta_0 \overset{d}{=} h_0$ which follows $AL(\theta,\kappa,\sigma)$. Hence the theorem.

The mean and variance of η_t are given by

$$\mathrm{E}(\eta_t) = (1-\rho)\left[\theta + \frac{\sigma}{\sqrt{2\left(\frac{1}{\kappa}-\kappa\right)}}\right] \text{ and } \mathrm{V}(\eta_t) = (1-\rho^2)\left[\frac{\sigma^2}{2\left(\frac{1}{\kappa}-\kappa\right)^2}+\sigma^2\right] \tag{10}$$

Hence the second order properties of the process $\{h_t\}$ are summarized below.

$$\mathrm{E}(h_t) = \theta + \frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right), \mathrm{V}(h_t) = \frac{\sigma^2}{2}\left(\frac{1}{\kappa^2}+\kappa^2\right), \text{and the ACF } \rho_{k(h_t)} = \rho^k, k = 1,2,\ldots \tag{11}$$

The regression of h_t on h_{t-1} is given by

$$\begin{aligned} E\left(h_t \middle| h_{t-1}\right) &= \rho h_{t-1} + E\left(\eta_t \middle| h_{t-1}\right) \\ &= \rho h_{t-1} + (1-\rho)\left[\theta + \frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right] \\ &= g\left(\theta,\kappa,\sigma;h_{t-1}\right). \end{aligned}$$

Next, we discuss the construction of a SV model generated by first order AL autoregressive process detailed in this section.

Asymmetric Laplace SV Model

Let $\{r_t\}$ be a sequence of returns on certain financial asset and the volatilities are generated by a Markov sequence $\{\exp(h_t)\}$ of non-negative random variables. Define the SV model

$$r_t = \exp\left(h_t/2\right)\varepsilon_t,$$

$$h_t = \rho h_{t-1} + \eta_t, \quad 0 \le \rho < 1, \ t = 1,2,\ldots,$$

where $\{\varepsilon_t\}$ is a sequence of independent and identically distributed standard Laplace random variables with mean zero and variance one. We assume that the sequence $\{\varepsilon_t\}$ is independent of h_t and η_t for every t. Here we assume that for every t, the volatility, h_t is an asymmetric Laplace random variables.

Since the sequence $\{\varepsilon_t\}$ follows standard Laplace distribution, the odd moments of r_t are zero and its even moments are given by

$$E\left(r_t^{2r}\right)=\frac{(2r)!\,e^{r\theta}}{2^r\left(1-\frac{1}{2}r^2\sigma^2-\frac{r\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right)},\ r=1,2,\ldots \tag{12}$$

Then

$$V\left(r_t\right)=\frac{e^{\theta}}{\left(1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right)} \tag{13}$$

and the kurtosis of r_t becomes

$$K=6\,\frac{\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]^2}{\left[1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]}.$$

By choosing different values for σ and κ, one can get a distribution with larger kurtosis as shown in Figure 1.

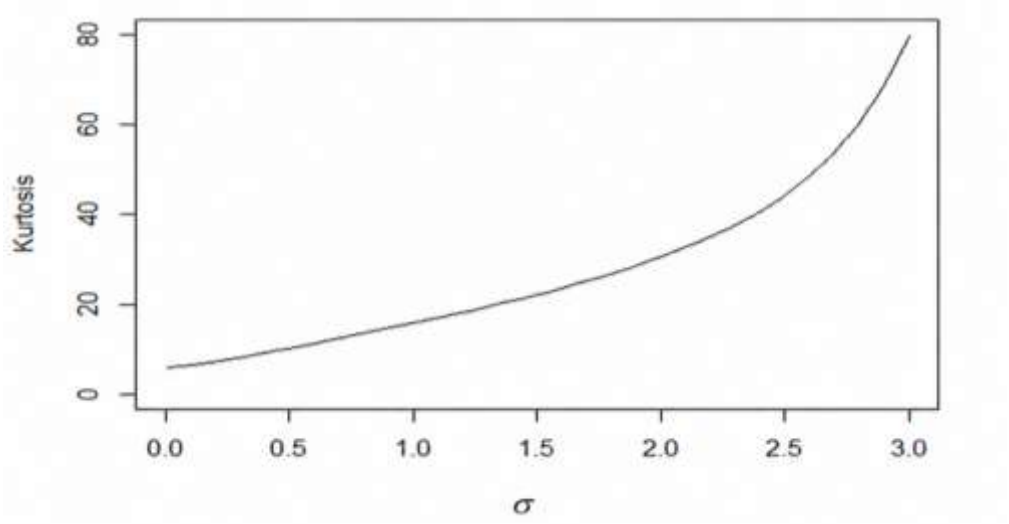

Figure 1. Kurtosis Pattern of the Proposed Return Distribution Under Different Volatility Parameter Values

The variance and covariance function of the squared return series are obtained as

$$V(r_t^2)=\frac{e^{2\theta}\left\{6\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]^2-\left[1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]\right\}}{\left[1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]^2},$$

$$Cov\left(r_t^2,\,r_{t-k}^2\right)=\frac{e^{2\theta}\sigma^2\rho^k\left[1+\frac{1}{2}\left(\frac{1}{\kappa}-\kappa\right)^2+\frac{\sigma^2\rho^k}{4}+\frac{\sigma}{2\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\left(1+\rho^k\right)\right]}{\left[1-\frac{\sigma^2\left(1+\rho^k\right)^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\left(1+\rho^k\right)\right]\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]^2}.$$

Hence, the lag k autocorrelation of the squared sequence $\left\{r_t^2\right\}$ is

$$\rho_k(r_t^2)=\frac{\sigma^2\rho^k\left[1+\frac{1}{2}\left(\frac{1}{\kappa}-\kappa\right)^2+\frac{\sigma^2\rho^k}{4}+\frac{\sigma}{2\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\left(1+\rho^k\right)\right]\left[1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]}{\left[1-\frac{\sigma^2\left(1+\rho^k\right)^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\left(1+\rho^k\right)\right]\left\{6\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]^2-\left[1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]\right\}}.$$

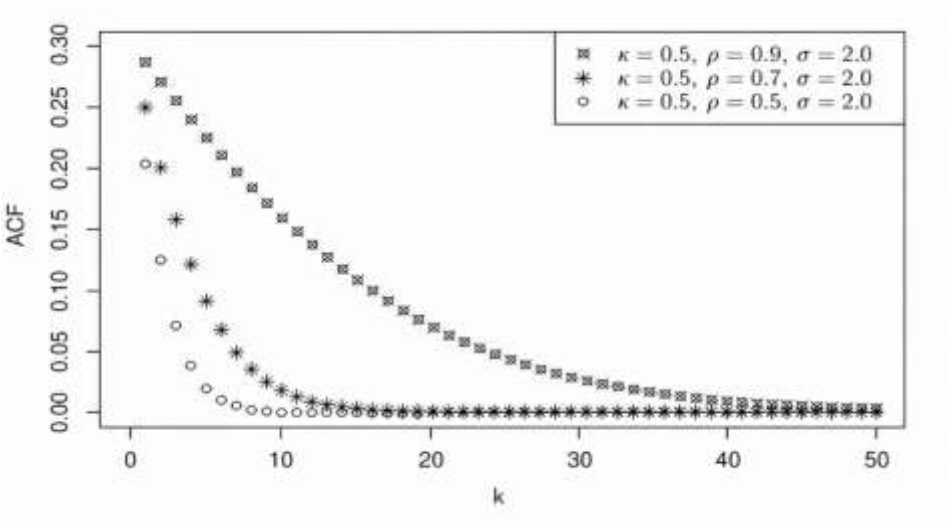

Figure 2. Autocorrelation Function of Squared Returns Under Different Parameter Combinations in the AL-SV Model

The ACF is an exponentially decreasing function of the lags for different values of the parameters, as can be seen in Figure 2. These results show that the model is capable of reproducing the long-lasting volatility persistence commonly observed in financial return series.

Parameter Estimation

One of the main difficulties in statistical inference for SV models is that the likelihood function depends on unobservable latent variables. These variables must be integrated out through multiple integrals, which complicates parameter estimation using the method of maximum likelihood. A wide range of alternative approaches to estimation and inference for parametric SV models has been proposed; reviews can be found in Broto and Ruiz (2004) and Tsay (2005). For the SV model described above, we adopt the generalized method of moments (GMM) to estimate the parameters.

Let $(r_1, r_2, ..., r_T)$ be a realization of length T from the AL-SV model and $\Theta=(\theta,\kappa,\sigma,\rho)'$ be the parameter vector to be estimated with true parameter value Θ_0. We use the moments

$$E\left(r_t^2\right)=\frac{e^{\theta}}{\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]},$$

$$E\left(r_t^4\right)=\frac{6e^{2\theta}}{\left[1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]},$$

$$E\left(r_t^6\right)=\frac{90e^{3\theta}}{\left[1-\frac{9}{2}\sigma^2-\frac{3}{\sqrt{2}}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]}$$

and

$$E\left(r_t^2\,r_{t-1}^2\right)=\frac{e^{2\theta}\left[1-\frac{\sigma^2\rho^2}{2}-\frac{\sigma\rho}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]}{\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]\left[1-\frac{\sigma^2(1+\rho)^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)(1+\rho)\right]}$$

to estimate the parameters.

Let $w_t = \left(r_t^2, r_t^4, r_t^6, r_t^2 r_{t-1}^2\right)'$ be a vector of random variables and $g\left(w_t, \theta\right)$ be a vector of functions defined as follows,

$$g\left(w_t, \theta\right) = \left(r_t^2 - E\left(r_t^2\right),\ r_t^4 - E\left(r_t^4\right), r_t^6 - E\left(r_t^6\right), r_t^2 r_{t-1}^2 - E\left(r_t^2 r_{t-1}^2\right)\right)'$$

Then the population moment condition takes the form $E\left[g\left(w_t, \Theta_0\right)\right] = 0$, for all t.

Method of moments involves choosing parameter estimators to minimize a quadratic form in a weighting matrix, W_T, and the sample moment $T^{-1}\sum_{t=1}^{T} g\left(w_t, \Theta\right)$. Then the GMM estimator based on this population moment condition is the value of Θ which minimizes,

$$Q_T\left(\Theta\right) = \left\{T^{-1}\sum_{t=1}^{T} g\left(w_t, \Theta\right)'\right\} W_T \left\{T^{-1}\sum_{t=1}^{T} g\left(w_t, \Theta\right)\right\}$$

where W_T is a positive definite weighting matrix. These equations have to be solved numerically. Let us denote by $\hat{\Theta}_M = \left(\hat{\theta}, \hat{\kappa}, \hat{\sigma}, \hat{\rho}\right)'$ the resulting estimators. Hansen (1982) proved that $\hat{\Theta}_M$ is consistent and asymptotically normal for Θ under the following regularity conditions:

A1. $\left\{r_t : -\infty < t < \infty\right\}$ is a stationary and ergodic sequence.

A2. The parameter space Θ is an open subset of R^q that contains the true parameter υ_0.

A3. $f\left(., \upsilon\right)$ and $\frac{\partial f}{\partial \upsilon}$ are Borel measurable for each $\upsilon \in \Theta$ and $\frac{\partial f\left(r, .\right)}{\partial \upsilon}$ is continuous on Θ for each $r \in R^q$.

A4. $\frac{\partial f_1}{\partial \upsilon}$ is first moment continuous at υ_0, $D = E\left[\frac{\partial}{\partial \upsilon} f\left(r_t, \upsilon_0\right)\right]$ exists, is finite, and has full rank.

A5. Let $\omega_t = f\left(r_t, \upsilon_0\right),\ -\infty < t < \infty$ and $\vartheta_j = E\left(\omega_0 \mid \omega_{-j}, \omega_{-j-1}, \ldots\right) - E\left(\omega_0 \mid \omega_{-j-1}, \omega_{-j-2}, \ldots\right),\ j \geq 0.$

The assumptions are that $E\left(\omega_0, \omega_0'\right)$ exists and is finite, $E\left(\omega_0 \mid \omega_{-j}, \omega_{-j-1}, \ldots\right)$ converges in mean square to zero and $\sum_{j=0}^{\infty} E\left(\vartheta_j' \vartheta_j\right)^{1/2}$ is finite.

These regularity conditions are readily verified for our model, and we have obtained the asymptotic dispersion matrix of the estimator. All the elements of asymptotic dispersion matrix are presented in Appendix.

Simulation Study

In this section, we carry out a Monte Carlo simulation study to evaluate the performance of the proposed estimators. We used the R package for computations, and the codes used are available with the authors. First, we generate a sample of size T from AL Markov sequence specified in (6) using the innovation random variable described in (8). Then simulate the sequence $\left\{r_t\right\}$ using (12) and (13). We present the following algorithm to generate $\left\{r_t\right\}$ from AL-SV model:

Specify values of the parameters $\left(\theta, \kappa, \sigma, \rho\right)'$ and for specified values of parameters generate exponential random variables $E_1 \sim Exp\left(\frac{\sigma}{\sqrt{2}\kappa}\right)$ and $E_2 \sim Exp\left(\frac{\sigma\kappa}{\sqrt{2}}\right)$.

1. Generate $\left(I_1, I_2\right)$ independent of E_1 and E_2 such that
   2. $P\left(I_1 = 0,\ I_2 = 0\right) = \rho^2,\ P\left(I_1 = 1,\ I_2 = 1\right) = \left(1-\rho\right)^2,\ P\left(I_1 = 1,\ I_2 = 0\right) = P\left(I_1 = 0,\ I_2 = 1\right) = \rho\left(1-\rho\right)$
3. Generate degenerate random variable U at $\theta\left(1-\rho\right)$.
4. Generate the $\left\{\eta_t\right\}$ using $\eta_t \overset{d}{=} U + \left(I_1 E_1 - I_2 E_2\right)$.

5. Simulate $\{h_t\}$ from the model $h_t = \rho h_{t-1} + \eta_t$; $0 \le \rho < 1$.
6. Generate $\{\varepsilon_t\}$ from standard Laplace distribution.
7. Simulate $\{r_t\}$ using the model $r_t = \exp(h_t/2)\varepsilon_t$ for $t = 1, 2, \ldots, n$.

Repeat steps 1 to 7 to get a sample of required size.

We use this simulated sample to obtain the estimates of the parameters by solving the moment equations given in the Section 5. For each specified value of parameter, we repeat the experiment 1000 times for computing the estimates and then averaged them over the repetitions. The average estimates and the corresponding RMSE based on the simulated samples are reported in Table 1 and 2.

Table 1. Average Moment Estimates and Mean Square Errors for Sample Size n = 1000 Under Different Parameter Values

ρ	σ	$\hat{\rho}$	$\hat{\sigma}$	$\hat{\kappa}$	$\hat{\theta}$
0.90		0.8650(0.1071)	0.4662(0.1835)	1.9626(0.1901)	0.9622(0.1715)
0.75		0.7112(0.1103)	0.4652(0.1752)	1.9659(0.2062)	0.9529(0.1866)
0.50	0.5	0.4701(0.1021)	0.4591(0.1854)	1.9512(0.2012)	0.9610(0.1696)
0.25		0.2296(0.1288)	0.4611(0.1752)	1.9588(0.2108)	0.9712(0.1811)
0.00		0.0421(0.1332)	0.4616(0.1899)	1.9521(0.2099)	0.9509(0.1885)
0.90		0.8710(0.1150)	0.9616(0.1899)	1.9550(0.2005)	0.9755(0.1785)
0.75		0.7132(0.1291)	0.9654(0.1821)	1.9688(0.2156)	0.9652(0.1755)
0.50	1.0	0.4735(0.1315)	0.9687(0.1921)	1.9569(0.2102)	0.9725(0.1792)
0.25		0.2345(0.1388)	0.9569(0.1825)	1.9491(0.2155)	0.9795(0.1804)
0.00		0.0302(0.1302)	0.9599(0.1847)	1.9545(0.2088)	0.9605(0.1808)
0.90		0.8720(0.1235)	1.4650(0.1665)	1.9478(0.2805)	0.9755(0.1865)
0.75		0.7129(0.1197)	1.4626(0.1769)	1.9560(0.2171)	0.9655(0.1711)
0.50	1.5	0.4713(0.1256)	1.4561(0.1864)	1.9589(0.2215)	0.9701(0.1865)
0.25		0.2360(0.1298)	1.4621(0.1902)	1.9510(0.2011)	0.9650(0.1715)
0.00		0.0592(0.1351)	1.4598(0.1892)	1.9529(0.2055)	0.9659(0.1895)

Table 2. Average Moment Estimates and Mean Square Errors for Sample Size n = 3000 Under Different Parameter Values

ρ	σ	$\hat{\rho}$	$\hat{\sigma}$	$\hat{\kappa}$	$\hat{\theta}$
0.90		0.8835(0.1015)	0.4782(0.1566)	1.9784(0.1821)	0.9805(0.1645)
0.75		0.7321(0.1078)	0.4693(0.1589)	1.9677(0.1845)	0.9869(0.1689)
0.50	0.5	0.4788(0.1101)	0.4745(0.1610)	1.9609(0.1894)	0.9798(0.1599)
0.25		0.2387(0.1178)	0.4792(0.1638)	1.9704(0.1812)	0.9840(0.1649)
0.00		0.0301(0.1278)	0.4823(0.1702)	1.9833(0.1904)	0.9799(0.1633)
0.90		0.8802(0.1067)	0.9788(0.1678)	1.9677(0.1789)	0.9809(0.1701)
0.75		0.7387(0.1137)	0.9756(0.1572)	1.9745(0.1862)	0.9769(0.1566)
0.50	1.0	0.4805(0.1089)	0.9721(0.1606)	1.9782(0.1890)	0.9788(0.1629)
0.25		0.2380(0.1024)	0.9820(0.1635)	1.9692(0.1798)	0.9846(0.1634)
0.00		0.0278(0.1178)	0.9788(0.1649)	1.9722(0.1893)	0.9766(0.1589)
0.90		0.8817(0.1123)	1.4820(0.1559)	1.9655(0.1972)	0.9809(0.1630)
0.75		0.7355(0.1078)	1.4783(0.1587)	1.9684(0.1867)	0.9840(0.1572)
0.50	1.5	0.4811(0.1200)	1.4845(0.1588)	1.9730(0.1793)	0.9895(0.1635)
0.25		0.2380(0.1189)	1.4734(0.1649)	1.9738(0.1827)	0.9741(0.1585)
0.00		0.0455(0.1278)	1.4751(0.1611)	1.9746(0.1845)	0.9793(0.1644)

From the above tables, we observe that the estimates are slightly biased. When the sample size is large, the estimators behave reasonably well and there is a significant reduction in bias of the estimates. Hence we claim that the GMM estimation yields reasonably good estimates for the parameters involved.

Empirical Analysis

To illustrate the application of the proposed model and the associated inferential results, we analyze two sets of financial data. The data sets used for this purpose are: (1) price (USD per barrel) of crude oil (West Texas Intermediate) futures from www.investing.com, with the data consisting of 1573 observations from January 02, 2020 to December 31, 2025; (2) daily 1-month forward premia (inter-bank) from www.rbi.org.in, with the data consisting of 2629 observations April 07, 2015 to March 30, 2026.

The time series plots of these data are given in Figure 3. While the original series exhibit non-stationarity, the return series fluctuates around zero and show periods of increased volatility and extreme movements. The presence of volatility clustering and large return shocks indicates the non-Gaussian return distribution for both the series.

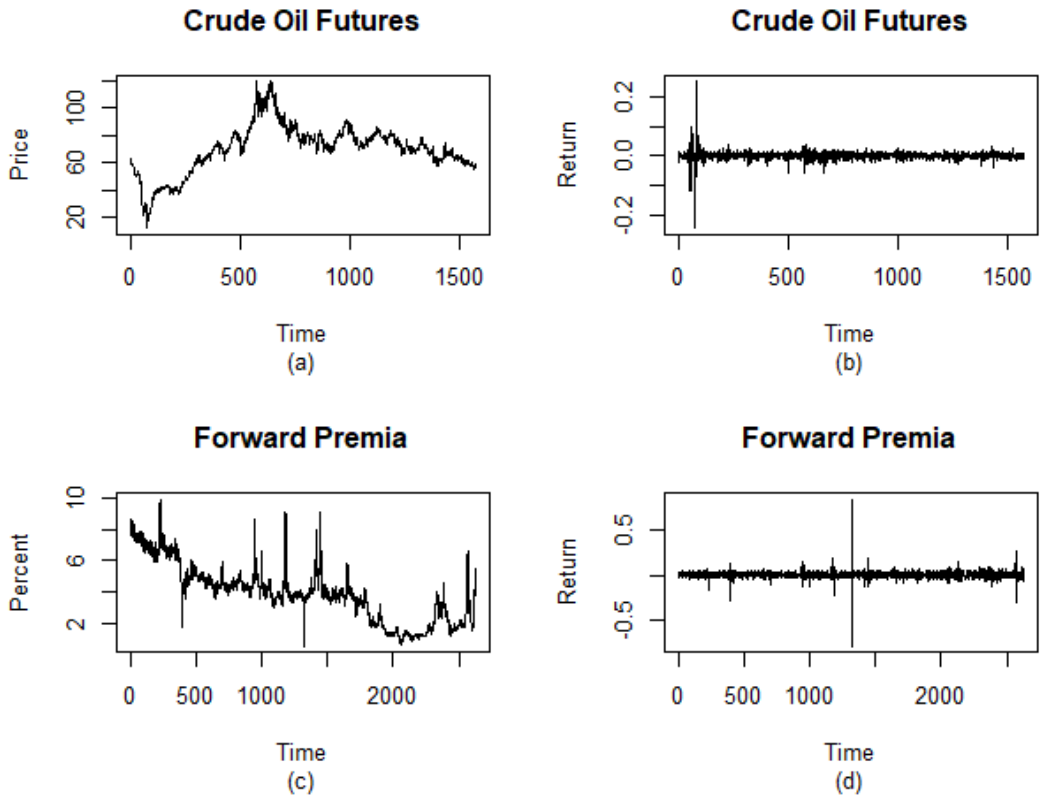

Figure 3. Panels (a) and (c) display the original time series of crude oil futures prices and forward premia, respectively, while panels (b) and (d) show the corresponding return series

Table 3 summarizes the descriptive statistics of the return series, including the mean, median, standard deviation, skewness, kurtosis and Jarque-Bera statistic. The return series are slightly skewed and the kurtosis is well above three, indicating that the return distribution is asymmetric and leptokurtic for both the series. The Jarque-Bera statistic is calculated for the test of joint hypothesis of zero skewness and excess kurtosis and statistic value clearly indicates the return data is non-normal. $Q(20)$ and $Q^{(2)}(20)$ are the Ljung-Box statistic for return and squared return series with lag 20. The corresponding χ^2 table value at 5% significance level is 10.117. Hence the test suggests that the return series is serially uncorrelated whereas the squared return series has significant serial correlation.

Table 3. Descriptive Statistics of Crude Oil Futures and Forward Premia Return Series Including Skewness, Kurtosis, and Jarque-Bera Values

Statistic	Exchange rate	Crude oil futures
Sample Size	1573	2629
Mean	0.0000	-0.0001
Minimum	-0.2469	-0.7852
Maximum	0.2524	0.8343
Std. Dev.	0.0151	0.0367
Skewness	-0.4793	0.5682
Kurtosis	102.5430	186.2971

$Q(20)$	3.2262	0.0723
$Q^{(2)}(20)$	11.7232	20.2069
Jarque-Bera	4971.2720	661.4428

From the ACF of the returns plotted in Figure 4, it is observed that serial correlations in the return series in the left panel are insignificant whereas the ACF of the absolute returns in the right panel are significant and declines with increasing lags very slowly.

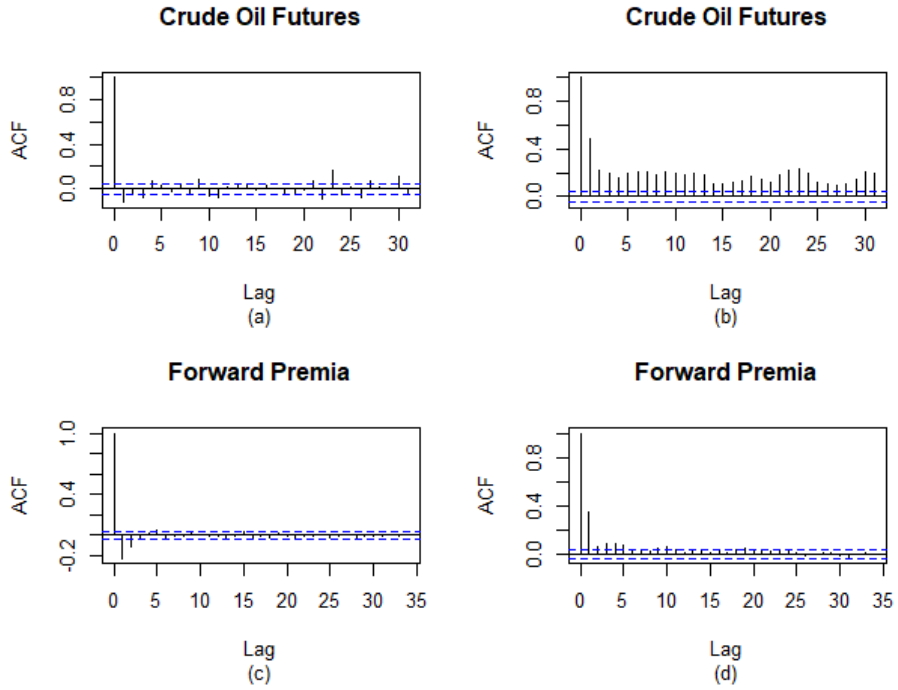

Figure 4. Panels (a) and (c) display the ACF of return series of crude oil futures prices and forward premia, respectively, while panels (b) and (d) show the ACF of corresponding absolute return series, indicating absence of serial correlation in return series and significant serial correlation in absolute return series

In Table 4, we present the parameter estimates for each of the return series. The values of the $\hat{\rho}$ in the Table suggest that there is a significant persistence of volatility in the above data series.

Table 4. Method-of-Moments Parameter Estimates for the AL-SV Model Applied to Crude Oil Futures and Forward Premia

Parameter	Exchange rate	Crude oil futures
$\hat{\theta}$	0.0003	-0.1030
$\hat{\kappa}$	0.5122	0.4971
$\hat{\sigma}$	0.2177	0.1076
$\hat{\rho}$	0.7810	0.8620

We now perform a diagnostic check of the model based on the residuals. For the AL-SV model, once the estimates of the parameters are obtained, the unobservable component $\{h_t\}$ is estimated using the Kalman filtering (for details see Jacquier *et al.* (1994), Tsay (2005)). We define the residuals of the model as $\hat{\varepsilon}_t = r_t \hat{h}_t^{-0.5}$, where $\hat{h}_t$ is the estimator of h_t provided by the Kalman filter at the MM estimate. If the fitted model is adequate, then $\{\hat{\varepsilon}_t\}$ should behave as an independent and identically distributed sequence of random variables with the assumed distribution. Since the model assumes that the residuals are independent, any dependence on either the residuals or their squares indicates misspecification of the model. In particular, if the fitted model is adequate, both series and should have no autocorrelations.

Figure 5 gives the sample autocorrelation of the residuals for the fitted AL-SV model. From the figures, the residuals appear to be random and their ACFs fail to indicate any significant serial dependence.

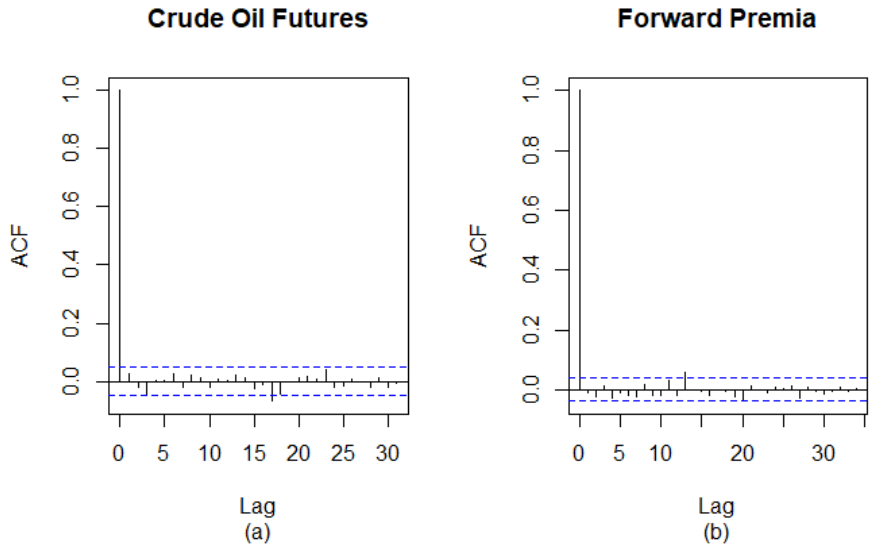

Figure 5. Sample ACF of residuals for (a) Crude oil futures prices and (b) Forward premia, indicating that no substantial serial correlation remains in the residuals

Further, we also checked the significance of ACF in the residues by computing the Ljung-Box statistic for the series $\{\hat{\varepsilon}_t\}$ and $\{\hat{\varepsilon}_t^2\}$, which are summarized in the Table 5. All these values are less than the 5% chi-square critical value (10.117) at degrees of freedom 19. Hence we conclude that there is no significant serial dependence among the residuals and the squared residuals.

Table 5. Ljung-Box Diagnostic Statistics for Residuals and Squared Residuals of the Fitted AL-SV Model

Data	Ljung-Box statistic	
	Residuals	Squared Residuals
Exchange rate	2.4065	0.0506
Crude oil futures	0.8789	0.2328

In the following Figure 6 we superimpose the histogram of the residuals with standard Laplace density to check whether the series follows standard Laplace distribution. The close correspondence between the empirical histograms and the theoretical density indicates that the standard Laplace distribution provides a satisfactory fit to the residuals in both cases.

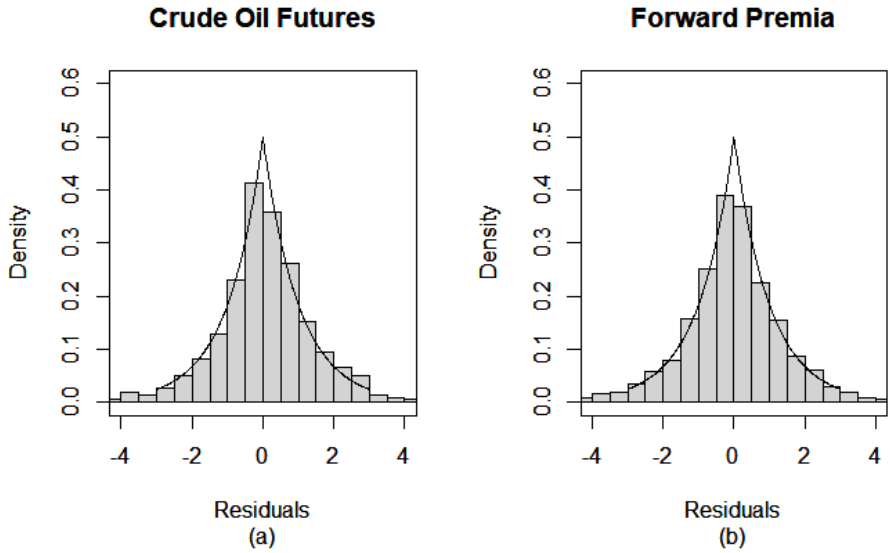

Figure 6. Histogram of residuals with superimposed standard Laplace density for (a) Crude oil futures prices and (b) Forward premia, indicates the standard Laplace distribution is a good fit for the residuals in both the cases.

Thus the data analysis illustrates that the proposed model is capable of capturing the stylized features of financial return series. These results suggest that the proposed SV model warrant further investigation in volatility forecasting, with forecasting performance evaluation and comparisons against existing models left for future research.

Conclusion

In this paper, the asymmetric Laplace distribution is used for modelling financial data, which exhibits asymmetry, sharp peaks and heavier tails than normal distribution. We proposed a stationary first order asymmetric Laplace autoregressive model to generate log-volatilities instead of Gaussian AR (1) model. Then we constructed a stochastic volatility model when the marginal distribution of log-volatility process has an asymmetric Laplace distribution, rather than the Gaussian distribution usually employed. The properties of the AL-SV model and its estimation procedures based on method of moments have been discussed along with a simulation study to evaluate the efficiency of the proposed estimation method. In the empirical study, we adopt AL-SV model to fit the daily returns of exchange rate and future price data. The data analysis illustrates that the AL-SV model is able to capture the skewness and excess kurtosis we observed in financial return series. Hence our proposed model is an alternative to existing SV models in the literature. The model presented here can be extended to analyze multivariate financial time series data. The family of multivariate AL laws can be obtained as a limiting case of the generalized hyperbolic distributions, introduced by Barndorff-Nielsen (1977) and seems to be suitable for modelling heavy tailed asymmetric multivariate data. However, the implementation of multivariate AL-SV model is not straight forward and work is currently under progress in this direction.

Acknowledgements

The authors thank the Reviewers and Editor-in-Chief for their constructive comments on an earlier version of the paper.

Conflicts of Interest

There are no conflicts to declare.

AI Writing Statement

The authors did not use any artificial intelligence-assisted technologies in the writing process.

References

Aas, K., & Haff, I. H. (2006). The generalized hyperbolic skew Student t-distribution. *Journal of Financial Econometrics*, *4*(2): 275–309.

Abraham, B., & Balakrishna, N. (1999). Inverse Gaussian autoregressive models. *Journal of Time Series Analysis*, *20*(6): 605–618.

Abraham, B., Balakrishna, N., & Sivakumar, R. (2006). Gamma stochastic volatility models. *Journal of Forecasting*, *25*(3): 153–171.

Adke, S. R., & Balakrishna, N. (1992). Markovian chi-square and gamma processes. *Statistics & Probability Letters*, *15*(5): 349–356.

Andersson, J. (2001). On the normal inverse Gaussian stochastic volatility model. *Journal of Business & Economic Statistics*, *19*(1): 44–54.

Anvar, P. M., Balakrishna, N., & Abraham, B. (2019). Stochastic volatility generated by product autoregressive models. *Stat*, *8*(1): e232.

Balakrishna, N. (2021). *Non-Gaussian autoregressive-type time series*. Singapore: Springer.

Balakrishna, N., & Nampoothiri, C. K. (2003). Cauchy autoregressive process and its applications. *Journal of Indian Statistical Association*, *41*(2): 143–156.

This journal article is © Thekkedath et al. 2026

Balakrishna, N., & Rahul, T. (2019). Modelling of stochastic volatility using Birnbaum-Saunders Markov sequence. *Statistics and Applications*, *17*(1): 105–120.

Balakrishna, N., & Shiji, K. (2014a). Extreme value autoregressive model and its applications. *Journal of Statistical Theory and Practice*, *8*(3): 460–481.

Balakrishna, N., & Shiji, K. (2014b). Stochastic volatility processes generated by Gumbel extreme value autoregressive model. *Journal of the Indian Statistical Association*, *52*(1): 45–64.

Barndorff-Nielsen, O. (1977). Exponentially decreasing distributions for the logarithm of particle size. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences*, *353*(1674): 401–419.

Barndorff-Nielsen, O. E. (1997). Normal inverse Gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics*, *24*(1): 1–13.

Bauwens, L., Hafner, C. M., & Laurent, S. (2012). *Handbook of volatility models and their applications*. John Wiley & Sons.

Bollerslev, T. (1986). Generalized autoregressive conditional heteroskedasticity. *Journal of Econometrics*, *31*(3): 307–327.

Broto, C., & Ruiz, E. (2004). Estimation methods for stochastic volatility models: A survey. *Journal of Economic Surveys*, *18*(5): 613–649.

Chib, S., Nardari, F., & Shephard, N. (2002). Markov chain Monte Carlo methods for stochastic volatility models. *Journal of Econometrics*, *108*(2): 281–316.

Choy, S. T. B., Wan, W. Y., & Chan, C. M. (2008). Bayesian Student-t stochastic volatility models via scale mixtures. *Advances in Econometrics*, *23*: 595–618.

Engle, R. F. (1982). Autoregressive conditional heteroscedasticity with estimates of the variance of United Kingdom inflation. *Econometrica*, *50*(4): 987–1007.

Gaver, D. P., & Lewis, P. A. W. (1980). First-order autoregressive gamma sequences and point processes. *Advances in Applied Probability*, *12*(3): 727–745.

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica*, *50*(4): 1029–1054.

Jacquier, E., Polson, N. G., & Rossi, P. (1994). Bayesian analysis of stochastic volatility models. *Journal of Business & Economic Statistics*, *12*(4): 371–389.

Jayakumar, K., & Kuttykrishnan, A. P. (2007). A time-series model using asymmetric Laplace distribution. *Statistics & Probability Letters*, *77*(16): 1636–1640.

Jose, K. K., & Thomas, M. M. (2011). Generalized Laplacian distributions and autoregressive processes. *Communications in Statistics - Theory and Methods*, *40*(23): 4263–4277.

Jose, K. K., Tomy, L., & Sreekumar, J. (2008). Autoregressive processes with normal-Laplace marginals. *Statistics & Probability Letters*, *78*(15): 2456–2462.

Kotz, S., Kozubowski, T., & Podgorski, K. (2001). *The Laplace distribution and generalizations: A revisit with applications to communications, economics, engineering, and finance*. Springer Science & Business Media.

Kozubowski, T. J., & Podgorski, K. (2000). A multivariate and asymmetric generalization of Laplace distribution. *Computational Statistics*, *15*(4): 531–540.

Lawrance, A. J., & Lewis, P. A. W. (1985). Modelling and residual analysis of nonlinear autoregressive time series in exponential variables. *Journal of the Royal Statistical Society. Series B*, *47*(2): 165–202.

Liesenfeld, R., & Jung, R. C. (2000). Stochastic volatility models: Conditional normality versus heavy-tailed distributions. *Journal of Applied Econometrics*, *15*(2): 137–160.

Nakajima, J. (2013). Stochastic volatility model with regime-switching skewness in heavy-tailed errors for

exchange rate returns. *Studies in Nonlinear Dynamics & Econometrics*, *17*(5): 499–520.

Nakajima, J., & Omori, Y. (2009). Leverage, heavy-tails and correlated jumps in stochastic volatility models. *Computational Statistics & Data Analysis*, *53*(6): 2335–2353.

Nakajima, J., & Omori, Y. (2012). Stochastic volatility model with leverage and asymmetrically heavy-tailed error using GH skew Student's t-distribution. *Computational Statistics & Data Analysis*, *56*(11): 3690–3704.

Omori, Y., Chib, S., Shephard, N., & Nakajima, J. (2007). Stochastic volatility with leverage: Fast and efficient likelihood inference. *Journal of Econometrics*, *140*(2): 425–449.

Popović, B. V. (2010). AR(1) time series with approximated beta marginal. *Publications de l'Institut Mathématique*, *80*(102): 89–98.

Popović, B. V., Pogány, T. K., & Nadarajah, S. (2010). On mixed AR(1) time series model with approximated beta marginal. *Statistics & Probability Letters*, *80*(19): 1551–1558.

Rahul, T., Balakrishnan, N., & Balakrishna, N. (2018). Time series with Birnbaum-Saunders marginal distributions. *Applied Stochastic Models in Business and Industry*, *34*(4): 562–581.

Shiji, K., & Rahul, T. (2022). On normal-Laplace stochastic volatility model. *Stochastics and Quality Control*, *37*(2): 127–136.

Sim, C. H. (1990). First-order autoregressive models for gamma and exponential processes. *Journal of Applied Probability*, *27*(2): 325–332.

So, M. K., Chu, A. M., Lo, C. C., & Ip, C. Y. (2022). Volatility and dynamic dependence modeling: Review, applications, and financial risk management. *Wiley Interdisciplinary Reviews: Computational Statistics*, *14*(5): e1567.

Sujith, P., & Balakrishna, N. (2023). Autoregressive inverse Gaussian process and the stochastic volatility modeling. *Communications in Statistics - Theory and Methods*, *52*(10): 3574–3580.

Taylor, S. J. (1986). *Modelling financial time series*. New York: John Wiley & Sons.

Tsay, R. S. (2005). *Analysis of financial time series*. New York: John Wiley & Sons.

Tsiotas, G. (2012). On generalised asymmetric stochastic volatility models. *Computational Statistics & Data Analysis*, *56*(1): 151–172.

Wang, J. J., Choy, S. T. B., & Chan, J. S. K. (2013). Modelling stochastic volatility using generalized t distribution. *Journal of Statistical Computation and Simulation*, *83*(2): 340–354.

Appendix: Asymptotic Dispersion Matrix

For computing the elements of the asymptotic dispersion matrix, the following observations become useful.

$$m_{2;2}^{(k)} = E\left(r_t^2 r_{t-k}^2\right) = \frac{E\left(e^{(\rho^k+1)h_{t-k}}\right)E\left(e^{h_t}\right)}{E\left(e^{\rho^k h_{t-k}}\right)};$$

$$m_{2;4}^{(k)} = E\left(r_t^2 r_{t-k}^4\right) = 6\,\frac{E\left(e^{(\rho^k+2)h_{t-k}}\right)E\left(e^{h_t}\right)}{E\left(e^{\rho^k h_{t-k}}\right)};$$

$$m_{2;6}^{(k)} = E\left(r_t^2 r_{t-k}^6\right) = 90\,\frac{E\left(e^{(\rho^k+3)h_{t-k}}\right)E\left(e^{h_t}\right)}{E\left(e^{\rho^k h_{t-k}}\right)};$$

$$m_{2;2,2}^{(k)} = E\left(r_t^2 r_{t-k}^2 r_{t-k-1}^2\right) = \frac{E\left(e^{(\rho^{k+1}+\rho+1)h_{t-k-1}}\right)E\left(e^{(\rho^k+1)h_{t-k}}\right)E\left(e^{h_t}\right)}{E\left(e^{(\rho^{k+1}+\rho)h_{t-k-1}}\right)E\left(e^{\rho^k h_{t-k}}\right)};$$

$$m_{4;2}^{(k)} = E\left(r_t^4 r_{t-k}^2\right) = 6\frac{E\left(e^{(2\rho^k+1)h_{t-k}}\right)E\left(e^{2h_t}\right)}{E\left(e^{2\rho^k h_{t-k}}\right)};$$

$$m_{4;4}^{(k)} = E\left(r_t^4 r_{t-k}^4\right) = 36\frac{E\left(e^{(2\rho^k+2)h_{t-k}}\right)E\left(e^{2h_t}\right)}{E\left(e^{2\rho^k h_{t-k}}\right)};$$

$$m_{4;6}^{(k)} = E\left(r_t^4 r_{t-k}^6\right) = 540\frac{E\left(e^{(2\rho^k+3)h_{t-k}}\right)E\left(e^{2h_t}\right)}{E\left(e^{2\rho^k h_{t-k}}\right)};$$

$$m_{4;2,2}^{(k)} = E\left(r_t^4 r_{t-k}^2 r_{t-k-1}^2\right) = 6\frac{E\left(e^{(2\rho^{k+1}+\rho+1)h_{t-k-1}}\right)E\left(e^{(2\rho^k+1)h_{t-k}}\right)E\left(e^{2h_t}\right)}{E\left(e^{(2\rho^{k+1}+\rho)h_{t-k-1}}\right)E\left(e^{2\rho^k h_{t-k}}\right)};$$

$$m_{6;2}^{(k)} = E\left(r_t^6 r_{t-k}^2\right) = 90\,\frac{E\left(e^{(3\rho^k+1)h_{t-k}}\right)E\left(e^{3h_t}\right)}{E\left(e^{3\rho^k h_{t-k}}\right)};$$

$$m_{6;4}^{(k)} = E\left(r_t^6 r_{t-k}^4\right) = 540\frac{E\left(e^{(3\rho^k+2)h_{t-k}}\right)E\left(e^{3h_t}\right)}{E\left(e^{3\rho^k h_{t-k}}\right)};$$

$$m_{6;2,2}^{(k)} = E\left(r_t^6 r_{t-k}^2 r_{t-k-1}^2\right) = 90\frac{E\left(e^{(3\rho^{k+1}+\rho+1)h_{t-k-1}}\right)E\left(e^{(3\rho^k+1)h_{t-k}}\right)E\left(e^{3h_t}\right)}{E\left(e^{(3\rho^{k+1}+\rho)h_{t-k-1}}\right)E\left(e^{3\rho^k h_{t-k}}\right)};$$

$$m_{2,2;2}^{(k)} = E\left(r_t^2 r_{t-1}^2 r_{t-k}^2\right) = \frac{E\left(e^{(\rho^k+\rho^{k-1}+1)h_{t-k}}\right)E\left(e^{(\rho+1)h_{t-1}}\right)E\left(e^{h_t}\right)}{E\left(e^{(\rho^k+\rho^{k-1})h_{t-k}}\right)E\left(e^{\rho h_{t-1}}\right)};$$

$$m_{2,2;4}^{(k)} = E\left(r_t^2 r_{t-1}^2 r_{t-k}^4\right) = 6\frac{E\left(e^{(\rho^k+\rho^{k-1}+2)h_{t-k}}\right)E\left(e^{(\rho+1)h_{t-1}}\right)E\left(e^{h_t}\right)}{E\left(e^{(\rho^k+\rho^{k-1})h_{t-k}}\right)E\left(e^{\rho h_{t-1}}\right)};$$

$$m_{2,2;6}^{(k)} = E\left(r_t^2 r_{t-1}^2 r_{t-k}^6\right) = 90\frac{E\left(e^{(\rho^k+\rho^{k-1}+3)h_{t-k}}\right)E\left(e^{(\rho+1)h_{t-1}}\right)E\left(e^{h_t}\right)}{E\left(e^{(\rho^k+\rho^{k-1})h_{t-k}}\right)E\left(e^{\rho h_{t-1}}\right)};$$

$$m_{2,2;2,2}^{(k)} = E\left(r_t^2 r_{t-1}^2 r_{t-k}^2 r_{t-k-1}^2\right) = \frac{E\left(e^{(\rho^{k+1}+\rho^k+\rho+1)h_{t-k-1}}\right)E\left(e^{(\rho^k+\rho^{k-1}+1)h_{t-k}}\right)E\left(e^{(\rho+1)h_{t-1}}\right)E\left(e^{h_t}\right)}{E\left(e^{(\rho^{k+1}+\rho^k+\rho)h_{t-k-1}}\right)E\left(e^{(\rho^k+\rho^{k-1})h_{t-k}}\right)E\left(e^{\rho h_{t-1}}\right)};$$

Where

$$E\left(e^{rh_t}\right) = \frac{e^{r\theta}}{1-\frac{\sigma^2 r^2}{2}-\frac{\sigma r}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)}.$$

Let $\Gamma_{(k)} = \begin{pmatrix} \gamma_{11}^{(k)} & \gamma_{12}^{(k)} & \gamma_{13}^{(k)} & \gamma_{14}^{(k)} \\ \gamma_{21}^{(k)} & \gamma_{22}^{(k)} & \gamma_{23}^{(k)} & \gamma_{24}^{(k)} \\ \gamma_{31}^{(k)} & \gamma_{32}^{(k)} & \gamma_{33}^{(k)} & \gamma_{34}^{(k)} \\ \gamma_{41}^{(k)} & \gamma_{42}^{(k)} & \gamma_{43}^{(k)} & \gamma_{44}^{(k)} \end{pmatrix}$, $k = 0, \pm 1, \pm 2, \ldots$

and $\Gamma_{(k)} = \Gamma_{(-k)}$, $k = 1, 2, \ldots$. Then the 4×4 matrix S is given by $S = \Gamma_{(0)} + 2\sum_{k=1}^{\infty}\Gamma_{(k)}$.

When $k = 0$, the elements of $\Gamma_{(0)} = E\left(\omega_t \omega_t^{'}\right)$ are obtained as

$$\gamma_{11}^{(0)} = m_{2;2}^{(0)} - c_1^2$$

$$\gamma_{12}^{(0)} = \gamma_{21}^{(0)} = m_{2;4}^{(0)} - c_1 c_2$$

$$\gamma_{13}^{(0)} = \gamma_{31}^{(0)} = m_{2;6}^{(0)} - c_1 c_3$$

$$\gamma_{14}^{(0)} = \gamma_{41}^{(0)} = m_{2;2,2}^{(0)} - c_1 c_4$$

$$\gamma_{22}^{(0)} = m_{4;4}^{(0)} - c_2^2$$

$$\gamma_{23}^{(0)} = \gamma_{32}^{(0)} = m_{4;6}^{(0)} - c_2 c_3$$

$$\gamma_{24}^{(0)} = \gamma_{42}^{(0)} = m_{4;2,2}^{(0)} - c_2 c_4$$

$$\gamma_{33}^{(0)} = m_{6;6}^{(0)} - c_3^2$$

$$\gamma_{34}^{(0)} = \gamma_{43}^{(0)} = m_{6;2,2}^{(0)} - c_3 c_4$$

$$\gamma_{44}^{(0)} = m_{2,2;2,2}^{(0)} - c_4$$

Similarly, the following are the elements of $\Gamma_{(k)}$ for $k = 1, 2, \ldots$

$$\gamma_{11}^{(k)} = m_{2;2}^{(k)} - c_1^2;\ \gamma_{12}^{(k)} = m_{2;4}^{(k)} - c_1 c_2;\ \gamma_{13}^{(k)} = m_{2;6}^{(k)} = c_1 c_3;\ \gamma_{14}^{(k)} = m_{2;2,2}^{(k)} - c_1 c_4 .$$

$$\gamma_{21}^{(k)} = m_{4;2}^{(k)} - c_1 c_2;\ \gamma_{22}^{(k)} = m_{4;4}^{(k)} - c_2^2;\ \gamma_{23}^{(k)} = m_{4;6}^{(k)} - c_2 c_3;\ \gamma_{24}^{(k)} = m_{4;2,2}^{(k)} - c_2 c_4 .$$

$$\gamma_{31}^{(k)} = m_{6;2}^{(k)} - c_1 c_3;\ \gamma_{32}^{(k)} = m_{6;4}^{(k)} - c_2 c_3;\ \gamma_{33}^{(k)} = m_{6;6}^{(k)} - c_3^2;\ \gamma_{34}^{(k)} = m_{6;2,2}^{(k)} - c_3 c_4 .$$

$$\gamma_{41}^{(k)} = m_{2,2;2}^{(k)} - c_1 c_4;\ \gamma_{42}^{(k)} = m_{2,2;4}^{(k)} - c_2 c_4;\ \gamma_{43}^{(k)} = m_{2,2;6}^{(k)} - c_3 c_4;\ \gamma_{44}^{(k)} = m_{2,2;2,2}^{(k)} - c_4^{\,2} .$$

The 4×4 matrix D is evaluated using the form $D=E\left(\frac{d}{d\upsilon}f\left(r_t,r_{t-1},\upsilon\right)\right)$ and its elements are:

$$D_{11}=-\frac{e^{\theta}}{1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)};\quad D_{12}=-\frac{12e^{2\theta}}{1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)};$$

$$D_{13}=-\frac{270e^{3\theta}}{1-\frac{9}{2}\sigma^2-\frac{3}{\sqrt{2}}\sigma\left(\frac{1}{\kappa}-\kappa\right)};\ D_{14}=-\frac{2e^{2\theta}\left[1-\frac{\sigma^2\rho^2}{2}-\frac{\sigma\rho}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]}{\left[1-\frac{\sigma^2(1+\rho)^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)(1+\rho)\right]\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]};$$

$$D_{21}=\frac{e^{\theta}\sigma\left(\frac{1}{\kappa^2}+1\right)}{\sqrt{2}\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]^2};\quad D_{22}=\frac{6\sqrt{2}e^{2\theta}\sigma\left(\frac{1}{\kappa^2}+1\right)}{\left[1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]^2};$$

$$D_{23}=\frac{270e^{3\theta}\sigma\left(\frac{1}{\kappa^2}+1\right)}{\sqrt{2}\left[1-\frac{9}{2}\sigma^2-\frac{3}{\sqrt{2}}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]^2}$$

$$D_{24}=-\frac{\frac{\sigma\rho e^{2\theta}}{\sqrt{2}}\left(\frac{1}{\kappa^2}+1\right)}{\left\{d_1(\sigma,\kappa,\rho)\ d_2(\sigma,\kappa,\rho)\right\}}+\frac{e^{2\theta}d_3(\sigma,\kappa,\rho)\left\{d_1(\sigma,\kappa,\rho)\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa^2}+1\right)+d_2(\sigma,\kappa,\rho)\frac{\sigma(1+\rho)}{\sqrt{2}}\left(\frac{1}{\kappa^2}+1\right)\right\}}{\left\{d_1(\sigma,\kappa,\rho)\ d_2(\sigma,\kappa,\rho)\right\}^2}$$

where,

$$d_1(\sigma,\kappa,\rho)=\left[1-\frac{\sigma^2(1+\rho)^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)(1+\rho)\right];$$

$$d_2(\sigma,\kappa,\rho)=\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right];\ d_3(\sigma,\kappa,\rho)=\left[1-\frac{\sigma^2\rho^2}{2}-\frac{\sigma\rho}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right].$$

$$D_{31}=\frac{e^{\theta}\left[\sigma+\frac{1}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]}{\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]^2},\ D_{32}=\frac{6e^{2\theta}\left[4\sigma+\sqrt{2}\left(\frac{1}{\kappa}-\kappa\right)\right]}{\left[1-2\sigma^2-\sqrt{2}\sigma\left(\frac{1}{\kappa}-\kappa\right)\right]^2},\ D_{33}=\frac{90e^{3\theta}\left[9\sigma+\frac{3}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]}{\left[1-\frac{9}{2}\sigma^2-\frac{3\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]^2},$$

$$D_{34}=-\frac{e^{2\theta}\left[\sigma\rho^2+\frac{\rho}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]}{\left\{d_1(\sigma,\kappa,\rho)\ d_2(\sigma,\kappa,\rho)\right\}}$$
$$+\frac{e^{2\theta}d_3(\sigma,\kappa,\rho)\left\{d_1(\sigma,\kappa,\rho)\left[\sigma+\frac{1}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]+d_2(\sigma,\kappa,\rho)\left[\sigma(1+\rho)^2+\frac{1}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)(1+\rho)\right]\right\}}{\left\{d_1(\sigma,\kappa,\rho)\ d_2(\sigma,\kappa,\rho)\right\}^2}$$

$$D_{41} = D_{42} = D_{43} = 0$$

$$D_{44} = -\frac{\left[1-\frac{\sigma^2(1+\rho)^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)(1+\rho)\right]\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]e^{2\theta}\left[-\sigma^2\rho-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]}{\left\{\left[1-\frac{\sigma^2(1+\rho)^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)(1+\rho)\right]\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]\right\}^2}$$
$$+\frac{e^{2\theta}\left[1-\frac{\sigma^2\rho^2}{2}-\frac{\sigma\rho}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]\left[\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]\left[-\sigma(1+\rho)-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]\right]}{\left\{\left[1-\frac{\sigma^2(1+\rho)^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)(1+\rho)\right]\left[1-\frac{\sigma^2}{2}-\frac{\sigma}{\sqrt{2}}\left(\frac{1}{\kappa}-\kappa\right)\right]\right\}^2}$$

Hence the asymptotic dispersion matrix becomes $\frac{1}{T}\Sigma$, where $\Sigma = \left[D S^{-1} D'\right]^{-1}$.