THE BETA TRANSMUTED POWER DISTRIBUTION: PROPERTIES AND APPLICATIONS*

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Abstract

In this this paper, we define and study a new generalization of the Power distribution and the quadratic rank transmutation map (QRTM) in order to generate a flexible family of probability distribution taking Power distribution as the base distribution. The new distribution is called the beta transmuted Power (BTP) distribution. Some properties of the distribution such as moments, quantiles, mean deviation and order statistics are derived. The method of maximum likelihood is proposed to estimate the model parameters. The asymptotic confidence intervals for the parameters are also obtained based on asymptotic variance-covariance matrix. A simulation study is conducted to study the performance of the estimators. The importance and flexibility of the new model is proved empirically using a real data set.

Keywords: beta power distribution, moments, parameter estimation, transmuted distribution.

1. Introduction

The power distribution is defined as the inverse of the Pareto distribution. Power function distribution is flexible lifetime distribution model which is the special case of beta distribution. Power function distribution was derived from Pareto distribution using the inverse transformation. According to Dallas (1976), if Y is power function distribution then Y^{-1} is the Pareto distribution model. Meniconi and Barry (1996) explore the performance of Power function distribution on electrical components and

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illustrated that power function distribution is most suitable distribution on electrical component data as compared to log-normal, Weibull and exponential models. Likewise, numerous probability models are used to model income distribution, but these models are mathematically more complicated to manage. The power function distribution on the other hand is very helpful in this regard, Naveed et. al. (2015). The power function distribution can be used to fit the distribution of likelihood ratios in statistical tests. The derivations of statistical properties of the power function distribution discussed by Johnson et. al. (1994), Balakrishnan and Nevzorov (2004), Kleiber and Kotz (2003) and Forbes et al. (2011). Characterizations of power function distribution using order statistics and record values has been studied by Ahsanullah (1973), Ahsanullah and Kabir (1974) discussed ordered statistics to estimate the scale and location of power function distribution. Zaka and Akhter (2014a), Zaka and Akhter (2014b) and Zaka et al. (2013) provided detailed discussion on parameter estimation of power function distribution using various estimation procedures like method of moments, maximum likelihood, percentiles, method of least square, and Bayesian estimation with various loss functions. Bayesian analysis of power function distribution was discussed using three single and as well as three double priors and the accuracy of these priors was assessed using simulation studies Sultan and Ahmad (2014). An initial test estimator for a scale parameter of the power function distribution was proposed by Sinha et al. (2008). Abdulsathar et al. (2015) estimate the Gini-index and Lorenz curve of power function distribution and the shape parameter using Bayesian approach. The estimators was developed using weighted squared error and squared error loss functions. Cordeiro and dos Santos Brito (2012) derived Beta power function, Tahir et al. (2014) introduced Weibull power function (WPF) distribution, and Oguntunde et al.(2015) studied the Kumaraswamy Power function distribution. Cumulative distribution function (cdf) and probability density function (pdf) of power function distribution is given by;

$$G(x) = \left(\frac{x}{\beta}\right)^{\alpha}, \qquad (1)$$

$$g(x) = \alpha\, \beta^{-\alpha}\, x^{\alpha-1}, \;\; 0 < x < \beta, \; \alpha > 0, \qquad (2)$$

where β is scale and α is shape parameter. A random variable X is said to have a transmuted Power probability distribution with parameter $\alpha > 0$ and $|\lambda| \leq 1$, if its pdf is given by

$$\mathrm{g(x)} = \frac{\alpha}{\beta^{\alpha}}\, x^{\alpha-1} \left(1 + \lambda - 2\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right), \qquad (3)$$

the corresponding cumulative distribution function is

$$\mathrm{G(x)} = \left(\frac{x}{\beta}\right)^{\alpha} \left(1 + \lambda - \lambda\left(\frac{x}{\beta}\right)^{\alpha}\right), \qquad (4)$$

where β is scale, α is shape parameter and λ is the transmuted parameter. A class of generalized distributions F(x) has been receiving considerable attention over the last few years, in particular, after the studies by Eugene, Lee, and Famoye (2002) and Jones (2004). If G denotes the baseline cumulative distribution function (cdf) of a random variable, then the beta generalized distribution is defined as

$$F(x) = I_{G(x)}(a,b) = \frac{1}{B(a,b)} \int_0^{G(x)} t^{a-1}\,(1-t)^{b-1} dt, \tag{5}$$

where $a > 0$ and $b > 0$ are shape parameters. Note that $I_y(a,b) = \frac{B_y(a,b)}{B(a,b)}$, is the incomplete beta function ratio, and $B_y(a,b) = \int_0^y t^{a-1}(1-t)^{b-1}\ dt$, is the incomplete beta function, $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function and $\Gamma(.)$ is the gamma function. The probability density function (pdf) of the Beta generated distribution has the form

$$\mathrm{f(x)} = \frac{\mathrm{g(x)}}{\mathrm{B}(a,\mathrm{b})}\ [\mathrm{G}(x)]^{\mathrm{a}-1}[1-\mathrm{G}(x)]^{\mathrm{b}-1}. \tag{6}$$

This class of generalized distribution has received considerable attention over the last years and several classical distributions have been generalized using this formulation. We generalize the transmuted Power distribution (3) using this formulation in order to construct the beta transmuted Power (BTP) distribution. We provide a comprehensive description of mathematical properties of BTP distribution and its application to analyze real data sets.

The rest of the paper is unfolded as follows. In Section 2 we define the BTP distribution and discuss some of its sub-models. In Section 3 we present the mixture representation of the BTP distribution. Section 4 discusses mathematical properties of the proposed family including ordinary moments, generating function, quantiles, mean deviation, order statistics and stress-strengh model. Estimation of parameters by the maximum likelihood method and performance of the estimators is assessed by simulation in Section 5. In Section 6, the distribution is used for analyzing real data. Finally, in Section 7, we make some concluding remarks on our study.

2. The Beta Transmuted Power Distribution

In this section, we provide the formulation of the beta transmuted Power (BTP) distribution. By inserting (4) into (5) the cumulative distribution function of the beta transmuted Power distribution with five parameters is given by

$$F(x) = I_{\left(\frac{x}{\beta}\right)^{\alpha}\ \left(1+\lambda-\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)}\quad (a,b)$$

$$= \frac{1}{B(a,b)} \int_0^{\left(\frac{x}{\beta}\right)^{\alpha}\ \left(1+\lambda-\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)} t^{a-1}\,(1-t)^{b-1} dt, \tag{7}$$

where $0 < \mathrm{x} < \beta,\ \alpha > 0,\ |\lambda| \le 1$ and $a > 0, b > 0$.

The cdf can be expressed in a closed form using the hypergeometric function (see Cordeiro and Nadarajah 2011) as follows:

$$F(x) = \frac{\left[\left(\frac{x}{\beta}\right)^{\alpha}\ \left(1+\lambda-\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^{a}}{aB(a,b)} .2F1\left(a, 1-b; a+1; \left(\frac{x}{\beta}\right)^{\alpha}\ \left(1+\lambda-\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)\right),$$

where $2F1(c,d;e;z) = \sum_{k=0}^{\infty} \frac{(c)_k\,(d)_k}{(e)_k} \frac{(z^k)}{k!}$ is the Gaussian hypergeometric function where $(c)_k$ is the ascending factorial defined by (assuming that $(c)_0 = 1$)

$$(\mathrm{c})_{\mathrm{k}} = \begin{cases} \mathrm{c}(\mathrm{c}+1)(\mathrm{c}+2)\ \ldots .\,(\mathrm{c}+\mathrm{k}-1) & \mathrm{k} = 1,2,3, \ldots \\ 1 & \mathrm{k} = 0 \end{cases}$$

Differentiating (7) with respect to x, we get the probability density function of the BTP distribution given by

$$f(x) = \frac{\alpha}{B(a,b)} \frac{x^{\alpha-1}}{\beta^{\alpha}} \left(1+\lambda - 2\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right) \left[\left(\frac{x}{\beta}\right)^{\alpha}\right]^{a-1} \left[\left(1+\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^{a-1} \left[1 - \left(\frac{x}{\beta}\right)^{\alpha}\left(1+\lambda-\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^{b-1}. \quad (8)$$

The beta transmuted Power (BTP) distribution includes the following distributions as special case:

- for $\lambda = 0$, beta transmuted Power reduces to beta Power distribution.
- For $a = b = 1$, beta transmuted Power reduces to transmuted Power distribution.
- For $a = b = 1$ and $\lambda = 0$, beta transmuted Power reduces to Power distribution.

Plots of the pdf (8) and the cdf (7) of beta transmuted Power distribution for some values of α, β, λ, a and b are given in Figures 1 and 2, respectively.

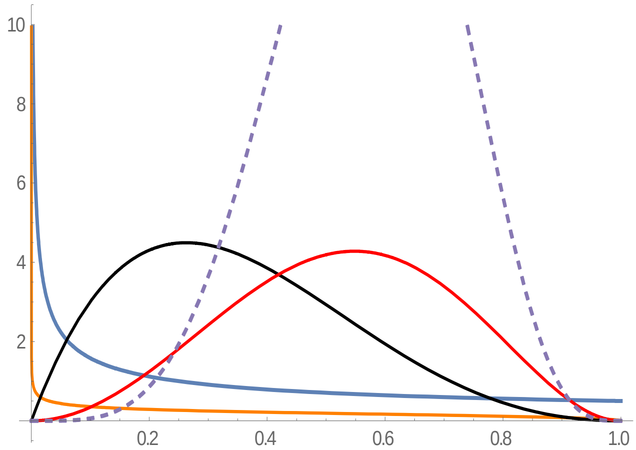

Figure 1: Pdf of beta transmuted Power distribution for $\beta = 1$ and (i) $\alpha = 0.5, \lambda = 0, a = b = 1$, (ii) $\alpha = 0.75, \lambda = 1, a = 2, b = 2$, (iii) $\alpha = 1, \lambda = 0.5, a = 2, b = 3$, (iv) $\alpha = 1.5, \lambda = 1, a = 2, b = 2$, (v) $\alpha = 2, \lambda = 1, a = 2.5, b = 3.5$.

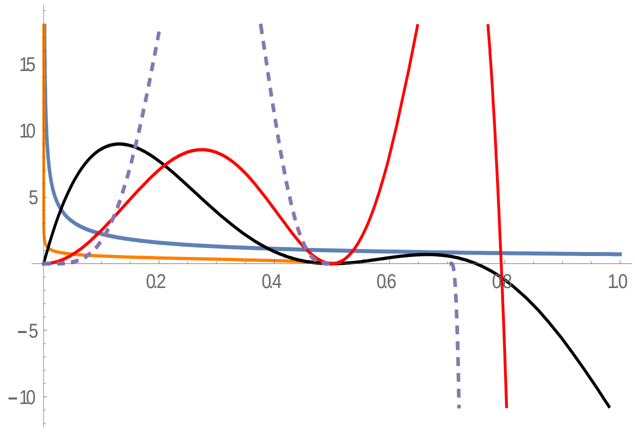

Figure 2: Cdf of beta transmuted Power distribution for $\beta = 0.5$ and (i) $\alpha = 0.5, \lambda = 0, a = b = 1$, (ii) $\alpha = 0.75, \lambda = 1, a = 2, b = 2$, (iii) $\alpha = 1, \lambda = 0.5, a = 2, b = 3$, (iv) $\alpha = 1.5, \lambda = 1, a = 2, b = 2$, (v) $\alpha = 2, \lambda = 1, a = 2.5, b = 3.5$.

3. Mixture Representation

In this section we find the series representations of the cdf and the pdf of the BTP distribution which will be useful to study its mathematical characteristics. As we shall see both pdf and cdf of BTP distribution can be expressed in terms of the Power distribution. By using (3) and the power series expansion of $(1-t)^{b-1}$, we get

$$\frac{1}{B(a,b)}\int_0^{G(x)} t^{a-1}(1-t)^{b-1}dt = \frac{1}{B(a,b)}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{j}\frac{[G(x)]^{a+j}}{(a+j)}$$

with the binomial term $\binom{b-1}{j} = \frac{\Gamma(b)}{\Gamma(b-j)\,j!}$ defined for any real b. Hence, (7) reduces to

$$F(x) = \sum_{i=0}^{\infty}(-1)^j\binom{b-1}{j}\frac{\left[\left(\frac{x}{\beta}\right)^{\alpha}\ \left(1+\lambda-\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^{a+j}}{B(a,b)(a+j)}. \quad (9)$$

Again, using the binomial expansion of $\left[\left(\frac{x}{\beta}\right)^{\alpha}\ \left(1+\lambda-\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)\right]^{a+j}$, we have

$$F(x) = \sum_{j,k,l=0}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{a+j}{k}\binom{a+j}{l}\lambda^{l}\frac{\left(\frac{x}{\beta}\right)^{\alpha(k+l)}}{B(a,b)(a+j)}$$

$$= \sum_{j,k,l=0}^{\infty}(-1)^{j+k}\binom{b-1}{j}\binom{a+j}{k}\binom{a+j}{l}\lambda^{l}\frac{\left(1-G_1(x;\alpha,\beta)\right)^{(k+l)}}{B(a,b)(a+j)} \quad (10)$$

where $\left(G_1(x;\alpha,\beta)\right)^{(k+l)}$ is the Power cdf with α,β parameter. Differentiating (10) with espect to x gives a useful expansion of $f(x)$ as

$$f(x) = \sum_{k,l=0}^{\infty} w_{kl}\ \left(g(x;\alpha,\beta)\right)^{(k+l)},\ 0 < x < \beta, \quad (11)$$

where

$$w_{kl} = \sum_{j=0}^{\infty} (-1)^{j+k+l} \binom{b-1}{j} \binom{a+j}{k} \binom{a+j}{l} \frac{\lambda^{l}}{B(a,b)(a+j)},$$

and $\left(g(x;\alpha,\beta)\right)^{(k+l)}$ is the Power pdf with α and β parameters.

4. Mathematical Characterizations

In this section we provide some mathematical properties of the BTP distribution including the moments, moment generating function, quantiles, mean deviations, order statistics and stress-strengh model.

4.1 Moments and moment generating function

Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., mean, dispersion, skewness and kurtosis). Using the mixture representation described in section 3, the r-*th* moment of the BTP random variable X is given by

$$E(X^r) = \int_{-\infty}^{\infty} x^r \, f(x) dx$$

$$= \int_0^{\beta} x^r \sum_{k,l=0}^{\infty} w_{kl} \left(f(x;\alpha,\beta)\right)^{(k+l)} dx$$

$$= \int_0^{\beta} \sum_{k,l=0}^{\infty} w_{kl} \, x^r \; \frac{\alpha}{\beta} \, (k+l) \left(\frac{x}{\beta}\right)^{\alpha(k+l)-1} dx$$

$$= \sum_{k,l=0}^{\infty} w_{kl} \sum_{r=0}^{\infty} \beta^r \frac{\alpha(k+l)}{\alpha(k+l)+r}. \tag{12}$$

The mean, variance, skewness, and kurtosis of the BTP are given by:

$$Mean = E(X) = \beta \sum_{k,l=0}^{\infty} w_{kl} \, \frac{\alpha(k+l)}{\alpha(k+l)+1} \tag{13}$$

$$Var(x) = \sum_{k,l=0}^{\infty} w_{kl} \left(\frac{(k+l)\,\alpha\,\beta^2}{(1+(k+l)\,\alpha)^2(2+(k+l)\,\alpha)}\right), \tag{14}$$

$$\text{Skewness}(x) \tag{15}$$
$$= \sum_{k,l=0}^{\infty} w_{kl} \left[\frac{-2(-1+(k+l)\,\alpha)\beta}{(1+(k+l)\,\alpha)(3+(k+l)\,\alpha)\sqrt{\frac{(k+l)\,\alpha\,\beta^2}{(1+(k+l)\,\alpha)^2(2+(k+l)\,\alpha)}}} \right],$$

$$Kurtosis(x) = \sum_{k,l=0}^{\infty} w_{kl} \left[9 + \frac{1}{(k+l)\alpha} + \frac{32}{3+(k+l)\alpha} - \frac{81}{4+(k+l)\alpha} \right] \tag{16}$$

Similarly, the moment generating function of X can be obtained as below:

$$M_X(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \sum_{k,l=0}^{\infty} w_{kl}\ (k+l)\,\alpha\,\beta^{-(k+l)\alpha}\,(-t)^{-(k+l)\alpha}\left[\Gamma\big((k+l)\alpha\big) - \Gamma\big((k+l)\alpha, -t\beta\big)\right]. \tag{17}$$

4.2 Quantiles

Quantiles are the points in a distribution that relates to the rank order of values. The quantile function of a distribution is the real solution of $F(x_q) = q$ for $0 \le q \le 1$. The quantiles of beta transmuted Power distribution are obtained from cdf (7) as

$$X = \beta\left[\left(1+\lambda+\sqrt{(1+\lambda)^2 - 4\lambda\left(I_q^{-1}(a,b)\right)}\right)/2\lambda\right]^{-\frac{1}{\alpha}}. \tag{18}$$

The following expansion for the inverse of the beta incomplete function $I_q^{-1}(a,b)$ can be found on the Wolfram website http://functions.wolfram.com/06.23.06.0004.01

$$I_u^{-1}(a,b) = w + \frac{b-1}{a+1}w^2 + \frac{(b-1)(a^2+3ab-a+5b-4)}{2(a+1)^2(a+2)}w^3 +$$
$$\frac{(b-1)[a^4+(6b-1)a^3+(b+2)(8b-5)a^2]}{2(a+1)^2(a+2)}w^4 + \frac{(b-1)[(33a^2-30b+4)a+b(31a-47)+18]}{3(a+1)^3(a+2)(a+3)}w^4 + O\left(P^{\frac{5}{a}}\right)$$

where $w = \{aB(a, b\,)q\}^{\frac{1}{a}}, a > 0.$

4.3 Mean Deviation

The amount of scatter in a population is evidently measured to some extent by the totality of deviations from the mean and median. If X has a BTP distribution, then we can derive the mean deviations about the mean $\mu = E(x)$ and about the median M as

$$\delta_1(x) = \int_0^{\beta} |x-\mu| f(x) dx,$$

and

$$\delta_2(x) = \int_0^{\beta} |x - M| f(x) dx.$$

The mean of the distribution is obtained from (13), and the median is obtained by solving the equation

$$I_{\left(\frac{x}{\beta}\right)^{\alpha}\left(1+\lambda-\lambda\left(\frac{x}{\beta}\right)^{\alpha}\right)}(a, b) = \frac{1}{2}.$$

These measures can be calculated using the relationships:

$$\delta_1(x) = \int_0^{\mu} (\mu - x) f(x) dx + \int_{\mu}^{\beta} (x - \mu) f(x) dx$$

$$= 2\int_0^{\mu} (\mu - x) f(x) dx = 2\left\{\mu F(\mu) - \int_0^{\mu} x f(x) dx\right\}$$

$$\delta_1(x) = 2\{\mu F(\mu) - J(\mu)\} \text{ and } \delta_2(x) = \mu - 2J(\mu), \quad (19)$$

where $J(t) = \int_0^t x f(x) dx$. From (11) we have

$$J(t) = \sum_{k,l=0}^{\infty} w_{kl} \int_0^t \alpha (k + l) \left(\frac{x}{\beta}\right)^{\alpha(k+l)} dx$$

$$J(t) = \sum_{k,l=0}^{\infty} w_{kl} \left(\frac{(k + l)\, \alpha\, t \left(\frac{t}{\beta}\right)^{\alpha(k+l)}}{1 + (k + l)\alpha}\right). \quad (20)$$

Using (10), one can easily find $\delta_1(x)$ and $\delta_2(x)$ from (19). The quantity J(t) can also be used to determine Bonferroni and Lorenz curves, which have applications in economics to study income and poverty, and also in other fields like reliability, demography, insurance and medicine. Bonferroni and Lorenz functions are given by $B(\pi) = \frac{J(p)}{(\pi\mu)}$ and $L(\pi) = \frac{J(p)}{\mu}$, respectively, where$p = Q(\pi)$ is calculated from (18) for a given probability π.

4.4 Order Statistics

Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ denote the order statistics in a data set from the BTP distribution with cumulative distribution function (7) and probability density function (8), then the pdf $f_{i:n}(x)$ of the *i*th order statistic $X_{(i)}$ is given by

$$f_{i:n}(x) = \frac{1}{B(i, n - i + 1)} f(x)[F(x)]^{i-1}[1 - F(x)]^{n-i},$$

where $\mathrm{i} = 1,2, \ldots, \mathrm{n}$, and the cdf is given by

$$F_{i:n}(x) = \sum_{k=i}^{n} \binom{n}{k} [F(x)]^k [1 - F(x)]^{n-k}$$

$$= \int_0^{F(x)} \frac{1}{B(i, n-i+1)} \, t^{i-1} [1-t]^{n-i} dt$$

Using expressions (10) and (11) for F(x) and f(x), respectively, and applying the binomial expansion, the above equation of the pdf $f_{i:n}(x)$ reduces to

$$\mathrm{f_{i:n}(x)} = \frac{1}{\mathrm{B(i, n-i+1)}} \, \mathrm{f(x)} \sum_{\mathrm{s}=0}^{\mathrm{n-i}} (-1)^{\mathrm{s}} \binom{\mathrm{n-i}}{\mathrm{s}} [\mathrm{F(x)}]^{\mathrm{i+s-1}}$$

$$f_{i:n}(x) = \frac{\alpha}{\mathrm{B(i, n-i+1)}} \left(\frac{1}{\beta} \sum_{\mathrm{k,l}=0}^{\infty} \mathrm{w_{kl}} \, (\mathrm{k} + \mathrm{l}) \left(\frac{\mathrm{x}}{\beta}\right)^{\alpha(\mathrm{k+l})-1} \right) \sum_{\mathrm{s}=0}^{\mathrm{n-i}} (-1)^{\mathrm{s+1}} \binom{\mathrm{n-i}}{\mathrm{s}} \left[\sum_{\mathrm{k,l}=0}^{\infty} \mathrm{w_{kl}} \left(\left(\frac{\mathrm{x}}{\beta}\right)^{\alpha(\mathrm{k+l})} \right) \right]^{\mathrm{i+s-1}} . \quad (21)$$

Writing $u = \left(\frac{\mathrm{x}}{\beta}\right)^{\alpha}$, $f_{i:n}(x)$ can be expressed as

$$f_{i:n}(x) = \frac{\alpha}{\mathrm{B(i, n-i+1)}} \frac{1}{\beta} \left(\sum_{\mathrm{k,l}=0}^{\infty} \mathrm{w_{kl}} \, (\mathrm{k} + \mathrm{l}) \, \mathrm{u}^{(\mathrm{k+l})-1} \right) \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \left[\sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \; u^{(k+l)} \right]^{i+s-1} . \quad (22)$$

We note that in (22) we can write

$$\sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \, u^{(k+l)} = \sum_{m=0}^{\infty} w_m^* \, u^m$$

and

$$\sum_{k,l=0}^{\infty} \mathrm{w}_{kl} \; (k+l) \, u^{(k+l)} = \sum_{m=0}^{\infty} m \, w_m^* \, u^m,$$

where $w_m^* \sum_{k,l:k+l=m} \mathrm{w}_{kl}$. Further, from Gradshteyn and Ryzhik (2000), for any positive integer r

$$\left(\sum_{k=0}^{\infty} \mathrm{a}_k \, u^k \right)^r = \sum_{k=0}^{\infty} d_{r,k} \, u^k, \quad (23)$$

where the coefficients $d_{r,k}$, for $k = 1,2,..,$ can be determined from the recurrence equation

$$d_{r,k} = \left(k_{a_0}\right)^{-1} \sum_{m=1}^{k} [m(r+1) - k]\, \mathrm{a}_m d_{r,k-m} \tag{24}$$

and $d_{r,0} = a_0^r$. Hence, $d_{r,k}$ comes directly from $d_{r,0}, \dots, d_{r,k-1}$ and, therefore, from $\mathrm{a}_0, \dots, \mathrm{a}_k$. Using (23) and (24) it follows that

$$f_{i:n}(x) = \frac{\alpha}{B(i, n-i+1)} \frac{1}{\beta} \left(\sum_{m=0}^{\infty} m\, w_m^* \, u^m \right) \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \left(\sum_{m=0}^{\infty} \mathrm{d}_{i+s-1,m} u^m \right),$$

where

$$\mathrm{d}_{i+s-1,m} = (m\, w_0^*)^{-1} \sum_{q=1}^{k} [q(i+s) - m] w_m^* \mathrm{d}_{i+s-1,m-q},$$

$$\mathrm{d}_{i+s-1,0} = (w_0^*)^{i+s-1} = \left(\sum_{j=0}^{\infty} (-1)^{j+1} \binom{b-1}{j} \frac{1}{B(a,b)(a+j)} \right)^{i+s-1} .$$

Combining terms, we obtain

$$f_{i:n}(x) = \frac{\alpha}{B(i, n-i+1)} \frac{1}{\beta} \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} m \mathrm{d}_{i+s-1,t^{w^*}t^{w^*}m} u^{m+t}$$

$$= \frac{1}{B(i, n-i+1)} \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} \frac{m \mathrm{d}_{i+s-1,t^{w_m^*}}}{m+t} \left[\frac{\alpha}{\beta} (m+t) \left(\frac{\mathrm{x}}{\beta} \right)^{\alpha(\mathrm{m+t})-1} \right]$$

$$f_{i:n}(x) = \sum_{m=l}^{\infty} \sum_{t=0}^{\infty} c_i(m,t)\;\; g(x{:}\,\beta, (m+t)\alpha), \tag{25}$$

where $g(x{:}\,\beta, (m+t)\alpha)$ denotes the pdf of a Power distribution with parameter β and $(m+t)\alpha$

$$c_i(m,t) = \frac{1}{B(i, n-i+1)} \frac{m\, w_m^*}{m+t} \sum_{s=0}^{n-i} (-1)^{s+1} \binom{n-i}{s} \mathrm{d}_{i+s-1,t}. \tag{26}$$

4.5 Stress-Strength Model

A stress-strength model describes the life of a component which has a random strength X_1 and is subjected to a random stress X_2. The component functions satisfactorily as long as $X_1 > X_2$, and fails when $X_1 < X_2$. The probability $R = Pr(X_1 > X_2)$ defines the component reliability. Stress-strength models have many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures and the aging of concrete pressure vessels.

Consider $X_1$and X_2 to be independently distributed, with $X_1 \sim BTP(\alpha_1, \beta, \lambda_1, a_1, b_1)$ and $X_2 \sim BTP(\alpha_2, \beta, \lambda_2, a_2, b_2)$. The cdf $F_1$of X_1 and pdf f_2 of X_2 obtained from (10) and (11), respectively. Then,

$$R = Pr(X_1 > X_2) = \int_0^{\beta} f_2(y)[1 - F_1(y)]dy$$

$$= 1 + \sum_{k,l=0}^{\infty} w_{kl}^{(1)} \int_0^{\beta} f_2(y) \left(\frac{y}{\beta}\right)^{\alpha(k+l)} dy$$

$$= \sum_{k,l=0}^{\infty} w_{kl}^{(1)} A(k,l),$$

where

$$w_{kl}^{(i)} = \sum_{j=0}^{\infty} (-1)^{j+k+l} \binom{b_i - 1}{j} \binom{a_i + j}{k} \binom{a_i + j}{l} \frac{\lambda^l}{B(a,b)(a_i + j)},$$

and

$$A(k,l) = \int_0^{\beta} f_2(y) \left(\frac{y}{\beta}\right)^{\alpha(k+l)} dy.$$

Now,

$$A(k,l) = \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \int_0^{\beta} (r+s) \frac{\alpha_2}{\beta} \left[\left(\frac{y}{\beta}\right)^{[\alpha_2(r+s)+\alpha_1(k+l)]-1}\right] dy$$

$$= \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \frac{\alpha_2(r+s)}{\alpha_1(k+l) + \alpha_2(r+s)}.$$

Hence,

$$R = 1 + \sum_{k,l=0}^{\infty} w_{kl}^{(1)} \sum_{r,s=0}^{\infty} w_{rs}^{(2)} \frac{\alpha_2(r+s)}{\alpha_1(k+l) + \alpha_2(r+s)}$$

$$= 1 + \sum_{k=0}^{\infty} \sum_{r=0}^{\infty} w_k^{*(1)} w_r^{*(2)} \frac{r\,\alpha_2}{k\,\alpha_1 + r\,\alpha_2}, \tag{27}$$

where

$$w_m^{*(i)} = \sum_{k,l:k+l=m} w_{kl}^{*(i)}, \quad i = 1,2.$$

5. Parameter Estimation

In this section we consider maximum likelihood estimation (MLE) to estimate the involved parameters. Asymptotic distribution of $\widehat{\Theta} = \left(\hat{\alpha}_i, \hat{\beta}_i, \hat{\lambda}_i, \hat{a}_i, \hat{b}_i\right)$ are obtained using the elements of the inverse Fisher information matrix.

5.1 Maximum Likelihood Estimation

In this section, we consider estimation by the method of maximum likelihood. Let $x_1, x_2, \dots, x_n$ be a random sample from the beta transmuted Power distribution with observed values $x_1, x_2, \dots, x_n$ and $\Theta = (\alpha, \beta, \lambda, a, b)$ be parameter vector. Then sample likelihood and Log-Likelihood functions of BTP is obtained as

$$L(\Theta) = \left(\frac{\alpha\,\beta^{-\alpha}}{B(a,b)}\right)^n \prod_{i=1}^{n} x_i^{\alpha-1} \left(1 + \lambda - 2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right) \left[\left(\frac{x_i}{\beta}\right)^{\alpha}\right]^{a-1} \left[\left(1 + \lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\right]^{a-1} \left[1 - \left(\frac{x_i}{\beta}\right)^{\alpha}\left(1 + \lambda - \lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\right]^{b-1}. \tag{28}$$

The log-likelihood function is

$$l(\Theta) = n\log\alpha - \alpha\, n\log\beta - n\log[B(a,b)] + (\alpha-1)\sum_{i=1}^{n} log\, x_i + \sum_{i=1}^{n} \mathrm{Log}\left[1 + \lambda - 2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right] + (a-1)\sum_{i=1}^{n} \mathrm{Log}\left(\frac{x_i}{\beta}\right)^{\alpha} + (a-1)\sum_{i=1}^{n} \mathrm{Log}\left(1 + \lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right) + (b-1)\sum_{i=1}^{n} \mathrm{Log}\left[\left(1 - \left(\frac{x_i}{\beta}\right)^{\alpha}\right)\left(1 + \lambda - \lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\right]. \tag{29}$$

We differentiate (29) with respect to $\alpha, \beta, \lambda, a$ and b respectively to obtain the elements of score vector $\frac{\partial l(\Theta)}{\partial \Theta} = \left(\frac{\partial l(\Theta)}{\partial \alpha}, \frac{\partial l(\Theta)}{\partial \beta}, \frac{\partial l(\Theta)}{\partial \lambda}, \frac{\partial l(\Theta)}{\partial a}, \frac{\partial l(\Theta)}{\partial b}\right)^T$ as below

$$\frac{\partial l(\Theta)}{\partial \alpha} = \frac{n}{\alpha} - n\log\beta + \sum_{i=1}^{n} \mathrm{Log}\, x_i + (a-1)\sum_{i=1}^{n}\log\left(\frac{x_i}{\beta}\right) + 2\lambda\sum_{i=1}^{n}\frac{\log\left(\frac{x_i}{\beta}\right)\left(\frac{x_i}{\beta}\right)^{\alpha}}{\left(1+\lambda-2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} + (a-1)\lambda\sum_{i=1}^{n}\frac{\log\left(\frac{x_i}{\beta}\right)\left(\frac{x_i}{\beta}\right)^{\alpha}}{\left(1+\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} - (b-1)\sum_{i=1}^{n}\frac{\lambda\log\left(\frac{x_i}{\beta}\right)\left(\frac{x_i}{\beta}\right)^{\alpha}\left(1-\left(\frac{x_i}{\beta}\right)^{\alpha}\right) + \log\left(\frac{x_i}{\beta}\right)\left(\frac{x_i}{\beta}\right)^{\alpha}\left(1+\lambda-\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)}{\left(1-\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\left(1+\lambda-\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} \quad (30)$$

$$\frac{\partial l(\Theta)}{\partial \beta} = -\frac{n\alpha}{\beta} - \frac{(a-1)n\alpha}{\beta} + \frac{2\alpha\lambda}{\beta^2}\sum_{i=1}^{n}\frac{x_i\left(\frac{x_i}{\beta}\right)^{\alpha-1}}{\left(1+\lambda-2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)}(a-1)\frac{\alpha\lambda}{\beta^2}\sum_{i=1}^{n}\frac{x_i\left(\frac{x_i}{\beta}\right)^{\alpha-1}}{\left(1+\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} + (b-1)\frac{\alpha}{\beta^2}\sum_{i=1}^{n}\frac{\lambda x_i\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(1-\left(\frac{x_i}{\beta}\right)^{\alpha}\right) + x_i\left(\frac{x_i}{\beta}\right)^{\alpha-1}\left(1+\lambda-\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)}{\left(1-\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\left(1+\lambda-\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)}, \quad (31)$$

$$\frac{\partial l(\Theta)}{\partial \lambda} = \sum_{i=1}^{n}\frac{\left(1-2\left(\frac{x_i}{\beta}\right)^{\alpha}\right)}{\left(1+\lambda-2\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} + (a-1)\sum_{i=1}^{n}\frac{\left(\frac{x_i}{\beta}\right)^{\alpha}}{\left(1+\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} + (b-1)\sum_{i=1}^{n}\frac{\left(1-\left(\frac{x_i}{\beta}\right)^{\alpha}\right)}{\left(1+\lambda-\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)} \quad (32)$$

$$\frac{\partial l(\Theta)}{\partial a} = -n[\Psi(a) - \Psi(a+b)] + \sum_{i-1}^{n}\mathrm{Log}\left(\frac{x_i}{\beta}\right)^{\alpha} + \sum_{i-1}^{n}\mathrm{Log}\left(1+\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right) \quad (33)$$

$$\frac{\partial l(\Theta)}{\partial b} = -n[\Psi(b) - \Psi(a+b)] + \sum_{i-1}^{n}\mathrm{Log}\left[\left(1-\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\left(1+\lambda-\lambda\left(\frac{x_i}{\beta}\right)^{\alpha}\right)\right], \quad (34)$$

where $\Psi(x)$ is the digamma function defined by $\Psi(x) = \frac{d\log\Gamma(x)}{dx}$, and $\Gamma(x)$ is the Gamma function. The solutions of nonlinear equations (30-34) are complicated to obtain, therefore an iterative procedure is applied to solve these equations numerically.

5.2 Asymptotic Distribution

We obtain the asymptotic distribution of $\widehat{\Theta} = (\hat{\alpha}_i, \hat{\beta}_i, \hat{\lambda}_i, \hat{a}_i, \hat{b}_i)$. The asymptotic variances of MLEs are given by the elements of the inverse of the Fisher information matrix. The Fisher information matrix of Θ, denoted by $J(\Theta) = E(I, \Theta)$, where $I_{ij}, i, j = 1,2,3,4,5$ is the observed information matrix. The second partial derivatives of the maximum likelihood function are given as the following:

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} & I_{15} \\ I_{21} & I_{22} & I_{23} & I_{24} & I_{25} \\ I_{31} & I_{32} & I_{33} & I_{34} & I_{35} \\ I_{41} & I_{42} & I_{43} & I_{44} & I_{45} \\ I_{51} & I_{52} & I_{53} & I_{54} & I_{55} \end{pmatrix}.$$

The exact mathematical expressions for $J(\Theta) = E(I, \Theta)$ are complicated to obtain. Therefore, the observed Fisher information matrix can be used instead of the Fisher information matrix. The variance-covariance matrix may be approximated as $V_{i,j} = I_{i,j}^{-1}$. The asymptotic distribution of the maximum likelihood can be written as follows (see Miller 1981).

$$[(\hat{\alpha} - \alpha), (\hat{\beta} - \beta), (\hat{\lambda} - \lambda), (\hat{a} - a), (\hat{b} - b)] \sim N_5(0, V). \qquad (35)$$

Since V involves the parameters $\alpha, \beta, \lambda, a\ and\ b$, we replace the parameters by the corresponding MLEs in order to obtain an estimate of V , which is denoted by $\hat{V}$. By using (35), approximate $100(1-\gamma)\%$ confidence intervals for $\alpha, \beta, \lambda, a\ and\ b$ are determined, respectively, as

$$\hat{\alpha} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{11}},\ \hat{\beta} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{22}},\ \hat{\lambda} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{33}},\ \hat{a} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{44}},\ \hat{b} \pm Z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{55}},$$

where Z_γ is the upper $100\gamma - th$ percentile of the standard normal distribution.

In the order to numerically illustrate the estimation of the involved parameters, we have simulated the ML estimators for different sample sizes $n = (10, 20, 50, 80, 100, 150, 300)$. The calculation of the estimation is based on 1000 simulated samples from the BTP. The MLEs and 95% confidence intervals are computed using the observed Fisher information matrix. Table 1 shows the average estimates, biases, standard errors (SE) and mean squared errors (MSE). In Table 2, the average 95% confidence limits (LCL & UCL) for the parameters $\alpha, \beta, \lambda, a\ and\ b$ are reported.

Table 1: Average of the estimates, bias, SE and MSE for BTP distribution.

N	Estimates	Bias	SE	MSE
10	$\hat{\alpha} = 2.29881$	0.43431	0.04157	0.18863
	$\hat{\beta} = 0.89005$	0.80758	0.37382	0.65219
	$\hat{\lambda} = 1.51930$	0.78028	0.05376	0.60884
	$\hat{a} = 3.10239$	0.95079	0.06238	0.90400
	$\hat{b} = 3.20428$	0.69279	0.08344	0.47996
20	$\hat{\alpha} = 2.33817$	0.41281	0.03826	0.05712
	$\hat{\beta} = 0.87941$	0.77197	0.34940	0.62721
	$\hat{\lambda} = 1.57805$	0.75500	0.05099	0.53062
	$\hat{a} = 3.14459$	0.92847	0.06006	0.87075
	$\hat{b} = 3.25189$	0.68519	0.05321	0.45436
50	$\hat{\alpha} = 2.36205$	0.40127	0.03526	0.01229
	$\hat{\beta} = 0.87381$	0.72128	0.31738	0.60867
	$\hat{\lambda} = 1.61311$	0.72006	0.04372	0.33647
	$\hat{a} = 3.16878$	0.90247	0.05457	0.82356
	$\hat{b} = 3.28095$	0.65761	0.02139	0.42808
80	$\hat{\alpha} = 2.36805$	0.32562	0.03283	0.00512
	$\hat{\beta} = 0.87249$	0.69312	0.28323	0.54904
	$\hat{\lambda} = 1.62186$	0.70813	0.04022	0.30795
	$\hat{a} = 3.17470$	0.88237	0.05017	0.76173
	$\hat{b} = 3.28827$	0.62711	0.01338	0.32132
100	$\hat{\alpha} = 2.37005$	0.29870	0.03014	0.00340
	$\hat{\beta} = 0.87207$	0.67932	0.25648	0.52916
	$\hat{\lambda} = 1.62478$	0.68176	0.03281	0.23844
	$\hat{a} = 3.17666$	0.86234	0.04413	0.42611
	$\hat{b} = 3.29072$	0.60695	0.01071	0.27190
150	$\hat{\alpha} = 2.37273$	0.22186	0.02617	0.00145
	$\hat{\beta} = 0.87150$	0.63303	0.21575	0.42329
	$\hat{\lambda} = 1.62867$	0.61823	0.03012	0.13910
	$\hat{a} = 3.17927$	0.82304	0.04145	0.12602
	$\hat{b} = 3.29399$	0.56873	0.00714	0.11600
300	$\hat{\alpha} = 2.37407$	0.14975	0.02317	0.00097
	$\hat{\beta} = 0.87122$	0.60933	0.19313	0.32941
	$\hat{\lambda} = 1.63061$	0.58261	0.02489	0.03943
	$\hat{a} = 3.18057$	0.76228	0.03177	0.02598
	$\hat{b} = 3.29563$	0.50866	0.00536	0.01447

From Table1 it is observed that when the sample size n increases, the MLE, Bias, SE and MSEs of parameters $\alpha, \beta, \lambda, a\ and\ b$ decrease. This verifies the consistency properties of the estimates.

Table 2: Average 95% confidence intervals for the parameters

n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{a}$	$\hat{b}$
10	(1.604, 1.619)	(0.621, 1.811)	(0.686, 0.747)	(0.298, 0.841)	(0.132, 0.915)
20	(1.583, 1.594)	(0.265, 1.453)	(0.698, 0.723)	(0.297, 0.758)	(0.223, 0.765)
50	(1.294, 1.322)	(0.496, 1.129)	(0.625, 0.649)	(0.433, 0.822)	(0.175, 0.610)
80	(1.292, 1.298)	(0.163, 1.157)	(0.664, 0.644)	(0.519, 0.786)	(0.221, 0.565)
100	(1.290, 1.296)	(0.756, 0.976)	(0.681, 0.642)	(0.518, 0.773)	(0.239, 0.547)
150	(1.289, 1.294)	(0.766, 0.954)	(0.611, 0.688)	(0.515, 0.749)	(0.268, 0.519)
300	(1.289, 1.292)	(0.569, 0.893)	(0.645, 0.634)	(0.526, 0.719)	(0.346, 0.482)

Table 2 shows that as the sample size increases, the average confidence lengths decrease and the intervals tend towards symmetry.

6. Application

In this section, we use real data set to compare the fits of the new model and illustrate the usefulness of the new model BTP and some of the models generated from Power distributions, namely: Power distribution (P), transmuted Power distribution (TP), and beta Power distribution (BP). The data set is obtained from Ghitany et al. (2008) consists of 100 observations on waiting time (in minutes) before the customer received service in a bank. The data are as follows: 0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8, 8.2, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7,10.9, 11, 11, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19, 19.9, 20.6,21.3, 21.4, 21.9, 23, 27, 31.6, 33.1, 38.5.

Table 3. Descriptive statistic of the Ghitany et. al. data

Min	1st. Qu.	Median	Mean	3rd. Qu.	Var.	Ske.	Kur.
0.8	4.65	8.1	9.877	13.05	52.374	1.473	5.540

In order to compare our results with other models. We estimate the parameters of the BTP model and compare its appropriateness to model this data with its sub-models including beta Power (BP), transmuted Power (TP) and Power (P) distributions.

The model selection is carried out by measuring the maximized log-likelihood (-2ℓ), the Akaike information criterion (AIC), the Bayesian information criterion (BIC), the consistent Akaike information criteria (CAIC) and the Hannan-Quinn information criterion (HQIC). Note that the smaller values of goodness-of-fit measures the better the fit of the data. These measures are defined as

$$AIC = 2K - 2l(\widehat{\Theta}),\ CAIC = AIC + \frac{2k(k+1)}{n-k-1},\ BIC = klog(n) - 2l(\widehat{\Theta}),$$

and

$$HQIC = 2klog\big(log(n)\big) - 2l(\widehat{\Theta}),$$

where $l(\widehat{\Theta})$ denotes the log-likelihood function evaluated at the maximum likelihood estimates, k is the number of parameters in the statistical model, n the sample size and Θ is the parameters. The, $\mathbf{-2\ell}$, AIC, BIC, $CAIC$ and $HQIC$ statistics for each model is provided in Table 3. It can be seen that BTP distribution leads to a better fit than any of its sub-models.

Table 4: The $\mathbf{-2\ell}$**,** AIC, CAIC, BIC and HQIC statistic to Ghitany data.

Distribution	$\mathbf{-2\ell}$	**AIC**	**CAIC**	**BIC**	**HQIQ**
BTP	436.1367	446.1367	446.7750	459.1626	451.4085
BP	427.5962	437.5962	438.2345	450.6221	442.8680
TP	423.7662	433.7662	434.4045	446.7920	439.0380
P	418.2667	428.2667	428.9050	441.2926	433.5385

The values of statistic to measure the goodness of the BTP distribution are provided in Table 4. To compare the BTP with beta Power (BP), transmuted Power (TP) and the Power (P) distributions. Since the values of the AIC, BIC, CAIC and HQIC are smaller for the BTP distribution compared with those values of the other models, the new distribution (BTP) could be chosen as the best model.

7. Concluding Remarks

In this paper, we have introduced the so-called beta transmuted Power (BTP) distribution. This is a generalization of the transmuted Power distribution using the genesis of the beta distribution. Many distributions including Power, beta Power and transmuted Power are embedded in this newly developed BTP distribution. For new generalization we derived its mathematical properties, explicit expressions for the moments, quantile function, generating functions and obtain the order statistics. We discuss estimation of the parameters by maximum likelihood and provide the information matrix. An application to a real data set indicates that the fit of the new model is superior to the fits of its principal sub-models.

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